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Finite-time stable estimator for attitude motion in the presence of bias in angular velocity measurements[☆]

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ABSTRACT

This article presents a nonlinear finite-time stable attitude estimation scheme for a rigid body with unknown dynamics and with unknown bias in angular velocity measurements. The attitude and angular velocity are estimated from a minimum of two linearly independent known vectors measured in the body-fixed frame, and the measured angular velocity vector is assumed to have a constant bias in addition to measurement errors. The estimated attitude evolves directly on the special orthogonal group SO(3) of rigid body rotations, avoiding any ambiguities and singularities. The constant bias in angular velocity measurements is also estimated. The estimation scheme is proven to be almost globally finite time stable in the absence of measurement errors using a Lyapunov analysis. The robustness of the scheme in the presence of bounded measurement errors is analytically shown. The estimation scheme is discretized as a geometric integrator for digital implementation. Numerical simulations demonstrate the finite time stability properties of the estimation scheme. Robustness of this estimation scheme is also demonstrated through a numerical comparison against some state-of-the-art nonlinear attitude estimation schemes.

1. Introduction

Estimation of attitude motion of a rigid body in three-dimensional Euclidean space is vital for a number of applications including unmanned aerial vehicles, spacecraft and underwater vehicles. The set of possible rotations of a rigid body is given by the set of 3×3 real orthogonal matrices of determinant 1, commonly called the Special Orthogonal group and denoted as SO(3). The nonlinear and compact attitude configuration space SO(3), makes the problem of attitude estimation an inherently nonlinear problem. As the attitude of the rigid body cannot be directly measured, the objective of an attitude estimator is to compute the orientation of the rigid body from vector measurements obtained from sensors mounted on the rigid body.

Attitude estimation has a long history, with early work like (Black, 1964; Wahba, 1965), proposing static attitude determination schemes. The performance of static determination schemes are often unsatisfactory in the presence of noise and bias components in measurements. Often, estimation schemes like modified Kalman Filter (Choukroun, Bar-Itzhack, & Oshman, 2006; Shuster, 1990) and Multiplicative Extended Kalman Filter (Markley, 1988) are used for attitude estimation. However, the implicit linearization in Kalman filter-like schemes may cause poor performance (Crassidis, Markley, & Cheng, 2007). More recent approaches have focused on nonlinear estimation schemes such as Bonnabel, Martin, and Rouchon (2009), Mahony, Hamel, and Pflimlin (2008) and Vasconcelos, Cunha, Silvestre, and Oliveira (2010), where the attitude estimate evolves on the nonlinear space SO(3) (or TSO(3), if angular velocity is also being estimated). Other prior work on nonlinear deterministic estimation schemes on SO(3) include Aguiar and Hespanha (2006), Barrau and Bonnabel (2017), Bonnabel et al. (2009), Hashim, Brown, and Mcisaac (2019), Lageman, Trumpf, and Mahony (2010), Mahony and Hamel (2017), Markley (2006), Moutinho, Figueirôa, and Azinheira (2015), Rehbinder and Ghosh (2003), Sanyal (2006) and Vasconcelos et al. (2010). Recent work on attitude observer on SO(3) with exponential stability are Gamagedara, Lee, and Chang (2019), in which the observer is developed with time varying reference directions, and Reis, Batista, Oliveira, and Silvestre (2018) that proposed an attitude observer based on single body-vector measurement. Attitude estimation schemes based on the Lagrange–d'Alembert principle from variational mechanics were first introduced in Izadi and Sanyal (2014) and subsequently developed in Izadi, Sanyal, Barany, and Viswanathan (2015a), Izadi, Sanyal, Beard, and Bai (2015b).

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Due to the topology of the compact manifold SO(3), no continuous attitude observer defined on the tangent bundle of SO(3) can provide convergence of the attitude estimation error to identity from all initial attitude and angular velocity estimation errors. This is shown in prior work like Bhat and Bernstein (2000) and Chaturvedi, Sanyal, McClamroch, et al. (2011). A continuous attitude observer at best can be almost global in terms of the region of attraction. For an attitude observer, almost global stability means that the attitude estimate stabilizes to the true attitude from almost all initial attitude estimates except those in a set of zero measure in the tangent space of SO(3). The attitude estimation scheme presented in Izadi, Viswanathan, Sanyal, Silvestre, and Oliveira (2016a) follows the variational framework of the estimation scheme reported in Izadi and Sanyal (2014), but includes bias in angular velocity measurements and estimates a constant bias vector. It is also shown that the proposed scheme is almost globally asymptotically stable, like the variational attitude estimator for the bias-free case.

In practice, the measured value of angular velocity often has bias. In the literature, separate schemes for bias estimation are employed to compensate for this bias. For example, in the estimation schemes provided in Izadi et al. (2016a), Mahony et al. (2008) and Tayebi, Roberts, and Benallegue (2011), an unknown constant bias is estimated along with the attitude. However, most of the proposed attitude estimation schemes and the bias estimation schemes are only asymptotically stable. There are advantages in having finite time stable estimation schemes: they have been shown to be more robust to disturbances and noise, and provide faster convergence than an asymptotically stable scheme with similar initial transience. Additionally, a finite time stable estimation scheme can automatically make a "separation principle" possible in case estimated state variables are used for feedback control. Finite time estimation schemes in the absence of bias using sliding mode controller and neural networks are proposed in Li, Wu, Shi, and Lim (2015), which are not continuous. Prior work by Bohn and Sanyal (2014) and Sanyal, Izadi, and Bohn (2014) proposed an almost global finite time stable attitude observer. However, the exact dynamical model including the moment of inertia is assumed to be available for estimation and the angular velocity bias was not considered. The algorithm in Warier, Sanyal, and Viswanathan (2019) provides finite time stable attitude estimation in the absence of bias without requiring the dynamics model of the rigid body.

This paper proposes an attitude estimation scheme with almost global finite time stability in the presence of constant angular velocity bias. The main contributions of the paper are: (1) the proposed attitude estimation scheme evolves on the special orthogonal group SO(3) and does not suffer from singularities or unwinding; (2) the estimation scheme is model-free in the sense that no assumptions are made on the attitude dynamics model including knowledge of the moment of inertia or the measurement noise model; (3) the estimation scheme is continuous and almost globally finite time stable in the absence of measurement errors even when the angular velocity measurement has an unknown constant bias; (4) the angular velocity bias estimate is also stabilized to the true value in finite time; (5) the robustness of the proposed scheme under time-varying noise in angular velocity measurements is analytically shown; and (6) the proposed algorithm is numerically compared with existing results from the literature including the discrete-time variational estimator (VAE) given in Izadi et al. (2016a), the geometric approximate minimum-energy (GAME) of Zamani, Trumpf, and Mahony (2011), and the constant gain observer (CGO) given in Mahony et al. (2008).

This work presents advancements over two of our prior publications: Warier et al. (2019), which provided a finite time stable attitude motion estimation scheme without any bias in measurements; and Sanyal, Warier, and Hamrah (2019), which provided preliminary results on the finite time stable attitude motion estimation scheme in the presence of bias in angular velocity measurements. The current manuscript adds to the paper (Warier et al., 2019) by providing an attitude motion estimator that is finite time stable when the angular velocity measurements have a constant bias. In addition, the bias in the angular velocity measurements is estimated in finite time. This work also adds to our preliminary work reported in the conference paper (Sanyal et al., 2019). The new additions here are: (a) in addition to the proof of finite-time stability in Sanyal et al. (2019), we explicitly prove the robustness of the proposed scheme under time-varying noise in angular velocity measurements; and (b) comparing the behavior of the proposed estimation scheme with some existing schemes from the literature that estimate attitude motion and angular velocity bias. The effectiveness of the proposed estimation scheme is then demonstrated by numerical simulations comparing its performance with that of the above-mentioned estimation schemes for a given attitude motion and a given set of measurements.

The structure of the paper is as follows. Section 2 outlines the mathematical notations and concepts used in the paper. The static attitude determination problem from vector measurements is posed in Section 3. Wahba's cost function is generalized by choosing a symmetric weight matrix and the resulting cost function is shown to be a Morse function on the Lie group of rigid-body rotations under some easy-to-satisfy conditions. Additionally, some useful lemmas associated with the cost function and its derivatives are provided. The real-time attitude and angular velocity bias estimation problem is detailed in Section 4. In Section 5, the estimation scheme is presented and finite time stability and robustness to angular velocity measurement errors are proved. Some existing state-of-the-art methods for attitude estimation are briefly introduced in Section 6. Numerical simulations of the finite-time stable attitude motion estimator with bias estimation, and comparisons of this estimator with the state-of-the-art nonlinear attitude observers of Section 6, are provided in Section 7. Finally, a summary of results and possible future directions are presented in Section 8.

2. Mathematical preliminaries

In this paper the set of real numbers are noted by $\mathbb{R}$. Similarly, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the set of real n-dimensional column vectors and real $n\times m$ matrices, respectively. $\mathbb{N}$ denotes the set of natural numbers. The set of all possible configurations of a rigid body is the special orthogonal group SO(3) (Murray, 2017), which is defined by:

$$\mathrm{SO}(3)=\left\{R\in\mathbb{R}^{3\times 3}|R^{\mathrm{T}}R=RR^{\mathrm{T}}=I,\ \det(R)=1\right\}.$$

This is a matrix Lie group under matrix multiplication. The Lie algebra (tangent space at identity) of SO(3) is denoted by $\mathfrak{so}(3)$ and is defined as:

$$\mathfrak{so}(3)=\left\{\mathrm{S}\in\mathbb{R}^{3\times 3}\mid \mathrm{S}=-\mathrm{S}^{\mathrm{T}}\right\},\ \mathrm{S}=\begin{bmatrix}0 & -s_3 & s_2\\ s_3 & 0 & -s_1\\ -s_2 & s_1 & 0\end{bmatrix}.$$

Let $(\cdot)^\times:\mathbb{R}^3\to\mathfrak{so}(3)$ denote the bijective map from three dimensional Euclidean space to $\mathfrak{so}(3)$. For a vector $s=[s_1\ s_2\ s_3]^{\mathrm{T}}\in\mathbb{R}^3$, the matrix $s^\times$ represents the vector cross product operator, that is $s\times r=s^\times r$, where $r\in\mathbb{R}^3$; this makes $(\cdot)^\times$ a vector space isomorphism. The inverse of $(\cdot)^\times$ is denoted by $\mathrm{vex}(\cdot):\mathfrak{so}(3)\to\mathbb{R}^3$, such that $\mathrm{vex}(a^\times)=a$, for all $a^\times\in\mathfrak{so}(3)$. We define the trace inner product on $\mathbb{R}^{m\times n}$ $\langle\cdot,\cdot\rangle$ as,

$$\langle A_1,A_2\rangle=\mathrm{tr}(A_1{}^{\mathrm{T}}A_2).$$

Any square matrix $A \in \mathbb{R}^{n\times n}$ can be written as sum of unique symmetric and skew-symmetric matrices as follows:

$$A = \mathrm{sym}(A) + \mathrm{skew}(A), \tag{1}$$

where the symmetric and skew-symmetric components are defined as,

$$\mathrm{sym}(A) = \frac{1}{2}(A + A^{\mathrm{T}}),\ \ \mathrm{skew}(A) = \frac{1}{2}(A - A^{\mathrm{T}}). \tag{2}$$

Additionally, following property holds. Let $A_1 \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $A_2 \in \mathbb{R}^{n\times n}$ be a skew symmetric matrix, then,

$$\langle A_1, A_2\rangle = 0. \tag{3}$$

In other words, symmetric and skew matrices are orthogonal under the trace inner product. For all $a_1, a_2 \in \mathbb{R}^3$,

$$\left\langle a_1^{\times}, a_2^{\times}\right\rangle = 2a_1 \cdot a_2 \tag{4}$$

With these definitions, we proceed to lay out the attitude estimation problem.

3. Static attitude determination from vector measurements

The aim of this section is to formulate the problem of attitude determination from vector measurements. Let $\mathcal{I}$ denote an inertial frame that is spatially fixed. A body-fixed frame is fixed to the rigid body with its origin at the center of mass of the body, and is denoted by $\mathcal{B}$. We denote the attitude of the rigid body by $R \in \mathrm{SO}(3)$, which transforms vectors in the body frame $\mathcal{B}$ to their counterparts in the inertial frame $\mathcal{I}$.

3.1. Vector measurements

The attitude of the rigid body is determined from body-fixed measurements of k known inertial vectors. Let $e_1, e_2, \ldots, e_k$, $k \in \mathbb{N}$ be the known inertial vectors and $u_1^m, u_2^m, \ldots, u_k^m$ be the corresponding body-fixed measurements. The ith vector measurement in the body-fixed frame $\mathcal{B}$ satisfies,

$$u_i^m = R^{\mathrm{T}} e_i + \sigma_i \tag{5}$$

where $\sigma_i \in \mathbb{R}^3$ is the noise in the ith vector measurement, for all $i \in 1, 2, \ldots, k$. The attitude of the rigid body can be calculated from the vector measurements provided the following assumption is satisfied.

Assumption 1. There are at least two non-collinear vectors in the set $\{e_1, \ldots, e_k\}$ for attitude determination at all times. If $k = 2$, $e_3 = e_1 \times e_2$ is selected as the third non-collinear vector.

Define the matrix consisting of k known inertial vectors e_i as column vectors,

$$E = \begin{cases} [e_1\ e_2\ e_1 \times e_2] \in \mathbb{R}^{3\times 3} & \text{when } k = 2, \\ [e_1\ e_2\ \ldots e_k] \in \mathbb{R}^{3\times k} & \text{when } k > 2. \end{cases} \tag{6}$$

Assumption 1 can be alternatively specified as follows: matrix E should have rank equal to 3. The corresponding matrix composed of body-fixed measurements as column vectors can be defined as,

$$U^m = \begin{cases} [u_1^m\ u_2^m\ u_1^m \times u_2^m] \in \mathbb{R}^{3\times 3} & \text{when } k = 2, \\ [u_1^m\ u_2^m\ \ldots\ u_k^m] \in \mathbb{R}^{3\times k} & \text{when } k > 2. \end{cases} \tag{7}$$

The matrix consisting of inertial vectors E and the matrix containing the body frame vectors U^m are related by:

$$U^m = R^{\mathrm{T}} E + \Xi, \tag{8}$$

where the columns of matrix Ξ correspond to the measurement errors σ_i. Let the true vectors in body frame be denoted by $u_i = R^{\mathrm{T}} e_i$, then the matrix of the actual body vectors corresponding to the inertial vectors e_i is given by

$$U = R^{\mathrm{T}} E, \tag{9}$$

in the absence of measurement errors.

The static attitude determination problem is formulated in the next subsection.

3.2. Cost function for attitude determination

The objective is to obtain an estimate of the attitude denoted by $\widehat{R} \in \mathrm{SO}(3)$ from k known inertial vectors $e_1, \ldots, e_k$ and corresponding measured vectors $u_1^m, \ldots, u_k^m$. The static attitude estimation can be formulated as an optimization problem as follows,

$$\mathrm{Minimize}_{\widehat{R}}\,\mathcal{U} = \frac{1}{2}\sum_{i}^{k} w_i (e_i - \widehat{R}u_i^m)^{\mathrm{T}}(e_i - \widehat{R}u_i^m), \tag{10}$$

where $w_i > 0$ are weight factors. This is referred to as Wahba's problem as in Wahba (1965). The cost function is re-expressed as,

$$\mathcal{U} = \frac{1}{2}\left\langle E - \widehat{R}U^m, (E - \widehat{R}U^m)W\right\rangle, \tag{11}$$

where $W = \mathrm{diag}([w_1,\ w_2,\ \ldots,\ w_k])$ and E and U^m are given by Eqs. (6) and (7) respectively. The cost function can be generalized such that W is a symmetric positive semi-definite matrix satisfying some special conditions. This is described in the next subsection. The structure of the generalized cost function in the absence of measurement errors, is detailed in the following lemma.

Lemma 1. *Define $Q = R\widehat{R}^T$ as the attitude estimation error. Let $E \in \mathbb{R}^{3\times k}$ be as defined as in (6) with rank$(E) = 3$. Let the gain matrix W of the generalized Wahba cost function be given by,*

$$W = E^T\left(EE^T\right)^{-1}K\left(EE^T\right)^{-1}E, \tag{12}$$

where $K = \mathrm{diag}([k_1, k_2, k_3])$ and $k_1 > k_2 > k_3 \geq 1$. Then, in the absence of measurement errors,

$$\mathcal{U} = \frac{1}{2}\left\langle E - \widehat{R}U^m, (E - \widehat{R}U^m)W\right\rangle = \langle K, I - Q\rangle, \tag{13}$$

which is a Morse function on SO(3) *whose critical points are given by the set,*

$$\mathcal{C} = \{I, \mathrm{diag}([-1, -1, 1]), \mathrm{diag}([1, -1, -1]), \\ \mathrm{diag}([-1, 1, -1])\}. \tag{14}$$

In addition, $\mathcal{U}$ has a global minimum at $Q = I$.

Proof. Utilizing the properties of the inner product we can arrive at the following simplification:

$$\mathcal{U} = \frac{1}{2}\mathrm{tr}\Big(\big(E^{\mathrm{T}}E + (U^m)^{\mathrm{T}}U^m\big)W\Big) \\ - \frac{1}{2}\mathrm{tr}\Big(\big((U^m)^{\mathrm{T}}\widehat{R}^{\mathrm{T}}E - E^{\mathrm{T}}\widehat{R}U^m\big)W\Big). \tag{15}$$

Substituting W as given by (12) and defining

$$L = EW(U^m)^{\mathrm{T}}, \tag{16}$$

the expression (15) can be further simplified to:

$$\mathcal{U} = \frac{1}{2}\mathrm{tr}\Big(K + U^m W (U^m)^{\mathrm{T}} - \widehat{R}^{\mathrm{T}}L - L^{\mathrm{T}}\widehat{R}\Big). \tag{17}$$

In the absence of measurement noise in the vector measurements, $U^m = U = R^{\mathrm{T}}E$ and $L = KR$. Substituting $U^m = R^{\mathrm{T}}E$ in Eq. (17) and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, we get

$$\mathcal{U} = \mathrm{tr}(K - L^{\mathrm{T}}\widehat{R}) = \mathrm{tr}\left(K - \widehat{R}R^{\mathrm{T}}EWE^{\mathrm{T}}\right) \\ = \langle K, I - Q\rangle,\ \text{where } Q = R\widehat{R}^{\mathrm{T}}. \tag{18}$$

The choice of W given in Eq. (12) ensures that $K = EWE^{\mathrm{T}}$. Next it is shown that $\langle K, I - Q\rangle$ is a Morse function with four isolated non-degenerate critical points on SO(3) given in (14). A Morse function is a function that has isolated non-degenerate critical points, which can be classified as minimum, maximum or saddle points by examining the Hessian of this function as in Milnor, Spivak, and Wells (1969). The proof that (13) is a Morse function is shown in Lemma 2.1 of Izadi and Sanyal (2014), and is omitted here for brevity.

A first variation of $Q \in \mathrm{SO}(3)$ is given by,

$$\delta Q = Q\Sigma^{\times}, \tag{19}$$

where $\Sigma \in \mathbb{R}^3$. The first variation of $\langle K, I - Q\rangle$ with respect to Q is given by

$$\partial_Q \langle K, I - Q\rangle = \langle K, -\delta Q\rangle = tr(-KQ\Sigma^{\times}). \tag{20}$$

KQ can be written as sum of skew and symmetric matrices i.e., $KQ = \mathrm{sym}(KQ) + \mathrm{skew}(KQ)$. Exploiting the linearity of the trace inner product and utilizing the identities given by Eqs. (3) and (4), the following expression is obtained.

$$\begin{aligned}\partial_Q \langle K, I - Q\rangle &= \langle \mathrm{skew}(KQ), \Sigma\rangle \\ &= \mathrm{vex}(KQ - Q^{\mathrm{T}}K)^{\mathrm{T}}\Sigma = s_K(Q)^{\mathrm{T}}\Sigma,\end{aligned} \tag{21}$$

where $s_K(Q)$ is given by,

$$s_K(Q) = \mathrm{vex}(KQ - Q^{\mathrm{T}}K), \tag{22}$$

and vex(.) is as defined in Section 2. The critical points of $\langle K, I - Q\rangle$ on SO(3) are where the variation vanishes. Since Σ is arbitrary, the critical points satisfy, $s_K(Q) = 0$, which implies,

$$KQ = Q^{\mathrm{T}}K. \tag{23}$$

Due to the properties of K, the critical points of $\langle K, I - Q\rangle$ are therefore given by,

$$\begin{aligned}Q \in \{&I, \mathrm{diag}([-1, -1, 1]), \mathrm{diag}([1, -1, -1]), \\ &\mathrm{diag}([-1, 1, -1])\}.\end{aligned} \tag{24}$$

By taking the second variation, it can be shown that $\langle K, I - Q\rangle$ achieves a minimum at $Q = I$. Similar results are available in prior literature, e.g., Bullo and Lewis (2004) and Izadi and Sanyal (2014). ■

The static estimation problem outlined here can be solved by computing $\widehat{R}$ that will minimize the $\mathcal{U}$ at any given instant. However, static methods often under perform when measurements have noise and bias. The following section considers dynamic attitude estimation under unknown attitude dynamics and with biased angular velocity measurements.

4. Preliminary results for attitude state and angular velocity bias estimation

4.1. Dynamic attitude estimation

The kinematics of rigid body rotation is given by Poisson's equation:

$$\dot{R} = R\Omega^{\times}, \tag{25}$$

where $\Omega \in \mathbb{R}^3$ is the true angular velocity of the rigid body represented in the body-fixed coordinate frame. Let the measured angular velocity, denoted by Ω^m, be given by

$$\Omega^m = \Omega + \beta + \nu, \tag{26}$$

where $\beta \in \mathbb{R}^3$ is a constant bias in angular velocity measurements that also has to be estimated, and $\nu \in \mathbb{R}^3$ is the vector of additive noise in angular velocity components. Let $(\widehat{R}, \widehat{\Omega}) \in \mathrm{SO}(3) \times \mathbb{R}^3$ be the estimated attitude and angular velocity states provided by the estimation scheme, satisfying the following kinematic relation:

$$\dot{\widehat{R}} = \widehat{R}\widehat{\Omega}^{\times}. \tag{27}$$

In addition, let

$$\widehat{\Omega} = \Omega^m - \widehat{\beta} - \omega, \tag{28}$$

where $\widehat{\beta} \in \mathbb{R}^3$ is the estimate of the bias in angular velocity measurements, and $\omega \in \mathbb{R}^3$ is the "excess" or error in estimating the angular velocity and the bias.

The objective is to obtain estimates of the attitude, angular velocity, and bias ($\widehat{R}$, $\widehat{\Omega}$ and $\widehat{\beta}$) in real time, from the matrix of known inertial vectors E, the corresponding vector measurements made in the body-fixed frame U^m, and the biased angular velocity measurement Ω^m. The moment of inertia and other parameters that occur in the dynamics of the rigid body are unknown. Note that the (number of) measured vectors may be varying over time, as long as at least two non-collinear vectors are measured at all times. The observer design given in the following section is shown to provide almost global finite-time stable (AGFTS) estimates $\widehat{R}$, $\widehat{\Omega}$ and $\widehat{\beta}$, where these estimates converge to the respective true values R, Ω and β in finite time, in the absence of measurement noise. Section 7 shows the robustness of this observer in the presence of measurement noise. The following result, relating the attitude estimation error to the angular velocity estimation error, is used in the next section to prove the main result.

Lemma 2. *Let K be as defined in Lemma 1. Then, in the absence of the measurement errors, the time derivative of $\mathcal{U}$ along the trajectories satisfying the kinematic equations (25)–(27), is given by:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U} &= \frac{\mathrm{d}}{\mathrm{d}t}\langle K, I - Q\rangle = s_K(Q)\cdot\big(\widehat{R}\widetilde{\Omega}\big) &(29)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}(K - L^T\widehat{R}) = -s_L(\widehat{R})\cdot\widetilde{\Omega}, &(30)\end{aligned}$$

where

$$\widetilde{\Omega} = \Omega - \widehat{\Omega}, \quad s_L(\widehat{R}) = \mathrm{vex}(L^T\widehat{R} - \widehat{R}^TL). \tag{31}$$

Proof. Since $Q = R\widehat{R}^{\mathrm{T}}$, we obtain from Eqs. (25)–(27):

$$\begin{aligned}\dot{Q} &= \frac{\mathrm{d}}{\mathrm{d}t}Q = \dot{R}\widehat{R}^{\mathrm{T}} + R\dot{\widehat{R}}^{\mathrm{T}} \\ &= R\Omega^{\times}\widehat{R}^{\mathrm{T}} - R\widehat{\Omega}^{\times}\widehat{R}^{\mathrm{T}} \\ &= R\widehat{R}^{\mathrm{T}}\big(\widehat{R}(\Omega - \widehat{\Omega})\big)^{\times} \\ &= Q(\widehat{R}\widetilde{\Omega})^{\times}.\end{aligned} \tag{32}$$

Further, from the definition of L in Eq. (16), we see that in the absence of measurement noise $U^m = U = R^{\mathrm{T}}E$ and

$$\dot{L} = EW\dot{U}^{\mathrm{T}} = EWU^{\mathrm{T}}\Omega^{\times} = L\Omega^{\times}. \tag{33}$$

From Eq. (32), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle K, I - Q\rangle = \langle K, -Q(\widehat{R}\widetilde{\Omega})^{\times}\rangle.$$

From Eq. (33), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}(K - L^{\mathrm{T}}\widehat{R}) = \mathrm{tr}(\Omega^{\times}L^{\mathrm{T}}\widehat{R} - L^{\mathrm{T}}\widehat{R}\widehat{\Omega}^{\times}).$$

As in the proof of Lemma 1, (3) and (4) are utilized to obtain,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle K, I - Q\rangle &= -\frac{1}{2}\mathrm{tr}\big((KQ - Q^{\mathrm{T}}K)(\widehat{R}\widetilde{\Omega})^{\times}\big) \\ &= \mathrm{vex}(KQ - Q^{\mathrm{T}}K)\cdot(\widehat{R}\widetilde{\Omega})\end{aligned} \tag{34}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}(K - L^{\mathrm{T}}\widehat{R}) = -\mathrm{vex}(L^{\mathrm{T}}\widehat{R} - \widehat{R}^{\mathrm{T}}L) \cdot \widetilde{\Omega}. \tag{35}$$

As (34) is identical to (29) and (35) is identical to (30), we conclude the result. ■

4.2. Some preliminary results

The following four lemmas are used to prove the main result on finite time stable attitude, angular velocity and bias estimation scheme given in Section 5.

Lemma 3. *Let x and y be non-negative real numbers and let $p \in (1, 2)$. Then*

$$x^{(1/p)} + y^{(1/p)} \geq (x + y)^{(1/p)}. \tag{36}$$

Moreover the above inequality is a strict inequality if both x and y are non-zero.

Proof. The proof of this result is given in Bohn and Sanyal (2014, 2016), and is omitted here for brevity. ■

Lemma 4. *Let K be as defined in Lemma 1 and $s_K(Q)$ be as given in Eq. (22). Let $\mathcal{S} \subset \mathrm{SO}(3)$ be a closed subset containing the identity in its interior, defined by*

$$\mathcal{S} = \big\{Q \in \mathrm{SO}(3) \,:\, Q_{ii} \geq 0 \text{ and } Q_{ij}Q_{ji} \leq 0 \quad \forall i,\ j \in \{1, 2, 3\},\ i \neq j\big\}. \tag{37}$$

Then for $Q \in \mathcal{S}$, we have

$$s_K(Q)^T s_K(Q) \geq \mathrm{tr}(K - KQ). \tag{38}$$

Proof. The proof of this result is given in Bohn and Sanyal (2016), and is omitted here for brevity. ■

Lemma 5. *Let $s_L(\widehat{R})$ and $s_K(Q)$ be as defined earlier. Then following holds:*

$$s_L(\widehat{R})^T s_L(\widehat{R}) = s_K(Q)^T s_K(Q) \tag{39}$$

Proof. From the definition of L, it can be seen that $L = KR$. $s_L(\widehat{R})$ and $s_K(Q)$ can be rewritten as,

$$s_L(\widehat{R}) = \mathrm{vex}(R^{\mathrm{T}}K\widehat{R} - \widehat{R}^{\mathrm{T}}KR) =: \mathrm{vex}(A_1), \tag{40}$$

$$s_K(Q) = \mathrm{vex}(KR\widehat{R}^{\mathrm{T}} - \widehat{R}R^{\mathrm{T}}K) := \mathrm{vex}(A_2), \tag{41}$$

where A_1, A_2 are used to represent the skew symmetric matrices inside the vex(.) operator. From the identity given in Eq. (4), it is clear that Eq. (39) is equivalent to following expression,

$$tr(A_1A_1) = tr(A_2A_2). \tag{42}$$

The RHS turns out to be,

$$tr(A_1A_1) = tr\big((R^{\mathrm{T}}K\widehat{R} - \widehat{R}^{\mathrm{T}}KR)(R^{\mathrm{T}}K\widehat{R} - \widehat{R}^{\mathrm{T}}KR)\big) \tag{43}$$

The LHS is obtained as,

$$tr(A_2A_2) = tr\big((KR\widehat{R}^{\mathrm{T}} - \widehat{R}R^{\mathrm{T}}K)(KR\widehat{R}^{\mathrm{T}} - \widehat{R}R^{\mathrm{T}}K)\big) \tag{44}$$

The identity in Eq. (42) can be obtained by expanding and simplifying the above two expressions using the properties of trace inner product. ■

The design and stability result of the finite-time stable estimator are given in the following section. Note that the attitude estimation error $Q = R\widehat{R}^{\mathrm{T}}$ is defined on the group of rigid body rotations, SO(3), which is not a vector space. The angular velocity estimation error, $\widetilde{\Omega}$, and bias estimation error, $\widetilde{\beta}$, are expressed on the vector space $\mathbb{R}^3$. Therefore, for Lyapunov stability analysis of the observer designed on $\mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3$, a suitable Lyapunov function is required. This comes in the form of a Morse–Lyapunov function, as defined later in Theorem 1 in Section 5, where the Morse function $\mathcal{U} = \langle K, I - Q\rangle$ on SO(3) is used as the component of the Morse–Lyapunov function that depends on the attitude component of the full state. The Morse–Lyapunov function is subsequently shown to guarantee convergence of state estimation errors $(Q, \widetilde{\Omega}, \widetilde{\beta})$ to $(I, 0, 0)$ in finite-time.

5. Finite-time stable attitude state and angular velocity bias estimation

In this section we give the main result; a finite-time stable observer for estimation of rigid body attitude, angular velocity, and a constant bias in angular velocity measurements. A Hölder-continuous Morse–Lyapunov function is utilized to show the finite-time stability of the resulting closed-loop system. For rigid body attitude, we assume that at least two non-collinear but known inertial vectors are measured in a body-fixed frame, as described earlier in Section 3.

Theorem 1. *Consider the attitude kinematics and angular velocity measurements given by equations (25)–(28) in the absence of measurement noise (i.e., $\sigma = 0$, $\nu = 0$). Let $p \in]1, 2[$ and $\kappa > 0$ be scalar observer gains, and define the following quantities:*

$$\widetilde{\beta} = \beta - \widehat{\beta}, \tag{45}$$

$$z_L(\widehat{R}) = \frac{s_L(\widehat{R})}{\big(s_L(\widehat{R})^T s_L(\widehat{R})\big)^{1-1/p}}, \tag{46}$$

$$\Psi(L, \widehat{R}, \omega) = \omega - \kappa z_L(\widehat{R}), \text{ and} \tag{47}$$

$$w_L(\widehat{R}, \widehat{\Omega}, \Omega^m, \widehat{\beta}) = \frac{\mathrm{d}}{\mathrm{d}t} s_L(\widehat{R}) = \mathrm{vex}\big(L^T\widehat{R}\widehat{\Omega}^\times + \widehat{\Omega}^\times\widehat{R}^T L\big) - \mathrm{vex}\big(\widehat{R}^T L(\Omega^m - \widehat{\beta})^\times + (\Omega^m - \widehat{\beta})^\times L^T\widehat{R}\big). \tag{48}$$

Let Assumption 1 be satisfied and let μ, k_p, k_v be positive scalar observer gains such that $\mu \leq 1$. Thereafter, consider the following observer equations:

$$\dot{\widehat{R}} = \widehat{R}\widehat{\Omega}^\times, \quad \text{where} \quad \widehat{\Omega} = \Omega^m - \widehat{\beta} - \omega, \quad \text{and} \tag{49}$$

$$\dot{\widehat{\beta}} = \frac{k_p}{2 - \mu} s_L, \quad \text{and} \tag{50}$$

$$\mu\dot{\omega} = k_p s_L - k_v \frac{\Psi}{\big(\Psi^T\Psi\big)^{1-1/p}} + \frac{\mu\kappa}{\big(s_L{}^T s_L\big)^{1-1/p}} H(s_L) w_L, \tag{51}$$

where the functional dependencies of s_L, w_L and Ψ have been suppressed for notational convenience, and where $H : \mathbb{R}^3 \to Sym(3)$, the space of symmetric 3×3 real matrices, is defined by

$$H(x) = I - \frac{2(1 - 1/p)}{x^T x} x x^T. \tag{52}$$

Then the attitude and angular velocity estimation errors (Q, ω) converge to $(I, 0) \in \mathrm{SO}(3) \times \mathbb{R}^3$ and the bias estimation error $\widetilde{\beta}$ converges to $0 \in \mathbb{R}^3$ in a finite time stable manner, from almost all initial conditions except those in a set of measure zero.

Proof. The purpose of this result is to obtain observer equations to estimate the attitude, angular velocity, and bias in real time, from the measured quantities U^m and Ω^m, assuming they do not have any noise. Since the angular velocity measurement, Ω^m, contains the effects of an unknown bias, β, as well as the true angular velocity, Ω, one can conclude $\Omega^m = \Omega + \beta$ in the absence of noise in angular velocity measurements. This justifies the use of $\Omega^m - \widehat{\beta}$ in place of the true angular velocity Ω in the observer equation (48).

Consider the following Morse–Lyapunov function:

$$\mathcal{V}(L, \widehat{R}, \Omega, \widehat{\Omega}, \widetilde{\beta}) = \frac{\mu}{2}\Psi^{\mathrm{T}}\Psi + k_p\, \mathcal{U}(\widehat{R}, U^m, E) + \frac{2-\mu}{2}\widetilde{\beta}^{\mathrm{T}}\widetilde{\beta}, \tag{53}$$

where $\mathcal{U}(\cdot,\cdot,\cdot)$ is as defined in Lemma 1. In the following analysis, we suppress the functional dependencies of $\mathcal{U}$ and $\mathcal{V}$ for notational ease. Taking the time derivative of this Lyapunov function, we get

$$\dot{\mathcal{V}} = \mu\Psi^{\mathrm{T}}\dot{\Psi} - k_p {s_L}^{\mathrm{T}}\widetilde{\Omega} - (2-\mu)\widetilde{\beta}^{\mathrm{T}}\dot{\widehat{\beta}}$$

where we used Eq. (30) to substitute for the time derivative of $\mathcal{U}$ in the second term, and the fact that β is a constant bias (therefore $\dot{\widetilde{\beta}} = -\dot{\widehat{\beta}}$) in the third term on the right side of the above expression. Substituting from Eq. (50) for $\dot{\widehat{\beta}}$ into the third term, we get

$$\dot{\mathcal{V}} = \mu\Psi^{\mathrm{T}}\dot{\Psi} - k_p {s_L}^{\mathrm{T}}(\widetilde{\Omega} + \widetilde{\beta}). \tag{54}$$

From Bohn and Sanyal (2014, 2016), we know that

$$\frac{d}{dt}z_L = \frac{1}{\left({s_L}^{\mathrm{T}}s_L\right)^{1-1/p}}H(s_L)w_L.$$

Now substituting for Ψ from Eq. (47) in to Eq. (54), using the time derivative of z_L as given by the above equation, and noting that $\omega = \widetilde{\Omega} + \widetilde{\beta}$ in the absence of measurement noise, we obtain

$$\dot{\mathcal{V}} = \mu\Psi^{\mathrm{T}}\left(\dot{\omega} - \frac{\kappa}{\left({s_L}^{\mathrm{T}}s_L\right)^{1-1/p}}H(s_L)w_L\right) - k_p{s_L}^{\mathrm{T}}\omega. \tag{55}$$

Finally, substituting the observer equation (51) for $\dot{\omega}$ into equation (55), we get

$$\begin{aligned}\dot{\mathcal{V}} &= \Psi^{\mathrm{T}}\left(k_p s_L - k_v\frac{\Psi}{\left(\Psi^{\mathrm{T}}\Psi\right)^{1-1/p}}\right) - k_p{s_L}^{\mathrm{T}}\omega \\ &= k_p{s_L}^{\mathrm{T}}(\Psi - \omega) - k_v\left(\Psi^{\mathrm{T}}\Psi\right)^{1/p} \\ &= -k_p\kappa{s_L}^{\mathrm{T}}z_L - k_v\left(\Psi^{\mathrm{T}}\Psi\right)^{1/p} \\ &= -k_p\kappa\left({s_L}^{\mathrm{T}}s_L\right)^{1/p} - k_v\left(\Psi^{\mathrm{T}}\Psi\right)^{1/p}.\end{aligned} \tag{56}$$

From Lemmas 4 and 5, in a neighborhood of $I \in \mathrm{SO}(3)$, we have

$$-{s_L}^{\mathrm{T}}s_L \le -\mathcal{U}(\widehat{R}, U^m, E) = -\langle K, I - Q\rangle. \tag{57}$$

Therefore, for the expression (56), we get

$$\begin{aligned}\dot{\mathcal{V}} &\le -k_p^{1-1/p}\kappa(k_p\mathcal{U})^{1/p} - k_v\left(\Psi^{\mathrm{T}}\Psi\right)^{1/p} \\ &\le -k_0\left(\left(\Psi^{\mathrm{T}}\Psi\right)^{1/p} + \left(k_p\mathcal{U}\right)^{1/p}\right),\end{aligned} \tag{58}$$

where $k_0 = \min(k_p^{1-1/p}\kappa, k_v)$.

Finally, applying Lemma 3 to the above inequality, we have:

$$\dot{\mathcal{V}} \le -k_0\left(\Psi^{\mathrm{T}}\Psi + k_p\mathcal{U}\right)^{1/p}. \tag{59}$$

Considering Eq. (59), the set where $\dot{\mathcal{V}} = 0$ is:

$$\begin{aligned}\dot{\mathcal{V}}^{-1}(0) =& \{(Q, \omega) \;:\; s_K(Q) = 0 \text{ and } \Psi = 0\} \\ =& \{(Q, \omega) \;:\; Q \in \mathcal{C} \text{ and } \omega = 0\}.\end{aligned} \tag{60}$$

where $\mathcal{C}$ is as defined by Eq. (14). Using the invariance-like theorem 8.4 in Khalil (2001), we can conclude that as $t \to \infty$, (Q, ω) converges to the set:

$$\mathsf{S} = \{(Q, \omega) \;:\; Q \in \mathcal{C} \text{ and } \omega = 0 \in \mathbb{R}^3\} \tag{61}$$

in finite time, which is equivalent to:

$$\begin{aligned}\mathsf{S} = \{&(Q, \widetilde{\Omega}, \widetilde{\beta}) \in \mathrm{SO}(3)\times\mathbb{R}^3\times\mathbb{R}^3 \\ &: \; Q \in \mathcal{C}\,,\ \widetilde{\Omega} = 0 \text{ and } \widetilde{\beta} = 0\},\end{aligned} \tag{62}$$

as $\omega = \widetilde{\Omega} + \widetilde{\beta}$, and Q satisfies the kinematics equation (32).

This means that the resulting observer system has a set of equilibria $\mathsf{S} \subset \mathrm{SO}(3)\times\mathbb{R}^3\times\mathbb{R}^3$, to which all initial estimation errors ultimately converge. Therefore, $\widetilde{\Omega}$ and $\widetilde{\beta}$ are ultimately bounded, and because $\omega = \widetilde{\Omega} + \widetilde{\beta}$ converges to the zero vector, the ultimate bound on $\|\widetilde{\beta}\|$ reflects the ultimate bound on $\|\widetilde{\Omega}\|$. In addition, since $\Psi = \omega - \kappa z_L(\widehat{R})$ is also ultimately bounded, and $s_K(Q)$, $s_L(\widehat{R})$, and therefore $z_L(\widehat{R})$ are also ultimately bounded, we can conclude that

$$c_l\,\Psi^{\mathrm{T}}\Psi \le \widetilde{\beta}^{\mathrm{T}}\widetilde{\beta} \le c_u\,\Psi^{\mathrm{T}}\Psi, \tag{63}$$

for positive constants c_l and c_u as time increases. This in turn allows us to conclude that

$$\begin{aligned}-\Psi^{\mathrm{T}}\Psi &= -(1-c_u)\,\Psi^{\mathrm{T}}\Psi - c_u\,\Psi^{\mathrm{T}}\Psi \\ &\le -(1-c_u)\,\Psi^{\mathrm{T}}\Psi - \widetilde{\beta}^{\mathrm{T}}\widetilde{\beta},\end{aligned} \tag{64}$$

on substituting Eq. (64) into Eq. (59) and applying Lemma 3 again, we conclude that

$$\dot{\mathcal{V}} \le -k_0\left((1-c_u)\,\Psi^{\mathrm{T}}\Psi + k_p\mathcal{U} + \widetilde{\beta}^{\mathrm{T}}\widetilde{\beta}\right)^{1/p}. \tag{65}$$

Finally, applying Theorem 7.1 and 7.2 in Bhat and Bernstein (2005), we conclude that the set of equilibria S given by Eq. (32) is finite-time stable. The only stable equilibrium in S is $(I, 0, 0)$ while the other three are unstable equilibria. The resulting closed-loop system with the estimation errors gives rise to a Hölder-continuous feedback with exponent less than one ($\frac{1}{p} < 1$), while in the limiting case of $\frac{1}{p} = 1$ the feedback system is Lipschitz-continuous. Proceeding with an analysis similar to that in Bohn and Sanyal (2014), Sanyal, Bohn, and Bloch (2013) and Sanyal et al. (2014), it can be concluded that the equilibria and the corresponding regions of attraction of the Hölder-continuous FTS observer with $p \in]1, 2[$ are identical to those of the corresponding Lipschitz-continuous asymptotically stable observer with $p = 1$, and the region of attraction is almost global. ■

5.1. Robustness analysis

The almost global finite-time stability property of the estimator given by Theorem 1 results in a guaranteed convergence of almost any bounded initial estimate errors to the true state, given by the estimation errors $(Q, \tilde{\Omega}, \tilde{\beta}) = (I, 0, 0)$, in finite time in the absence of any disturbances. In the presence of a bounded measurement noise ν in the measurement of angular velocity, all estimate errors will converge to a bounded neighborhood of $(I, 0, 0)$. The following result gives a conservative statement relating the bound of measurement noise that can be tolerated and bounds on the neighborhood of $(I, 0, 0)$.

Corollary 1. *Consider the observer equations (49)–(51). Let the measured angular velocity be given by*

$$\Omega^m = \Omega + \beta + \nu \tag{66}$$

where ν is the time-varying noise vector. Let $\mathcal{N} \subset S\times\mathbb{R}^3\times\mathbb{R}^3$, where S is as defined in Eq. (37), be a closed neighborhood of $(I, 0, 0)$ defined by

$$\mathcal{N} := \left\{(Q, \tilde{\Omega}, \tilde{\beta}) : \|s_L\| \le s_{L_{max}} \text{ and } \|\Psi\| \le \Psi_{max} < 1\right\}. \tag{67}$$

If the norm of the noise in angular velocity ν satisfies following inequality,

$$\|\nu(t)\| \leq \epsilon \leq \frac{k_v\left(s_{L_{max}}^{(2/p)}+\Psi_{max}^{(2/p)}\right)}{k_p\ s_{L_{max}}}, \tag{68}$$

then, the estimation errors $(Q, \tilde{\Omega}, \tilde{\beta})$ converge to the neighborhood $\mathcal{N}$.

Proof. The proof of this statement is based on the Lyapunov analysis used in the proof of Theorem 1. Substituting (26) in (28), we find

$$\tilde{\Omega}+\tilde{\beta}=\omega-\nu, \tag{69}$$

where $\tilde{\Omega}$ and $\tilde{\beta}$ are defined in (31) and (45), respectively. Substituting (69) into the time derivative of Lyapunov functions that is obtained and simplified as in (54), we have

$$\dot{\mathcal{V}}=\mu\Psi^{\mathrm{T}}\dot{\Psi}-k_p{s_L}^{\mathrm{T}}\omega+k_p{s_L}^{\mathrm{T}}\nu, \tag{70}$$

Then substituting for Ψ from (47) into (70) using the time derivative of z_L, in the presence of measurement noise, we obtain

$$\dot{\mathcal{V}}=\mu\Psi^{\mathrm{T}}\left(\dot{\omega}-\frac{\kappa}{\left({s_L}^{\mathrm{T}}s_L\right)^{1-1/p}}H(s_L)w_L\right) \tag{71}$$
$$-k_p{s_L}^{\mathrm{T}}\omega+k_p{s_L}^{\mathrm{T}}\nu.$$

Finally, substituting the expression given in (51) for $\dot{\omega}$ into (71), we get

$$\begin{aligned}\dot{\mathcal{V}}&=\Psi^{\mathrm{T}}\left(k_ps_L-k_v\frac{\Psi}{\left(\Psi^{\mathrm{T}}\Psi\right)^{1-1/p}}\right)-k_p{s_L}^{\mathrm{T}}\omega+k_p{s_L}^{\mathrm{T}}\nu\\&=k_p{s_L}^{\mathrm{T}}(\Psi-\omega)-k_v\left(\Psi^{\mathrm{T}}\Psi\right)^{1/p}+k_p{s_L}^{\mathrm{T}}\nu\\&=-k_p\kappa{s_L}^{\mathrm{T}}z_L-k_v\left(\Psi^{\mathrm{T}}\Psi\right)^{1/p}+k_p{s_L}^{\mathrm{T}}\nu\\&=-k_p\kappa\left({s_L}^{\mathrm{T}}s_L\right)^{1/p}-k_v\left(\Psi^{\mathrm{T}}\Psi\right)^{1/p}+k_p{s_L}^{\mathrm{T}}\nu.\end{aligned} \tag{72}$$

which has an additional term due to the measurement noise, when compared with expression (56) for $\dot{\mathcal{V}}$ along the noise-free observer. Considering upper bounds of the noise as defined in (68) on this extra term, we find

$$k_p{s_L}^{\mathrm{T}}\nu\leq\|k_p\|\ \|s_L\|\ \|\nu\|\leq k_ps_{L_{max}}\epsilon \tag{73}$$

Therefore, $\dot{\mathcal{V}}$ is upper bounded as

$$\dot{\mathcal{V}}\leq-k_p\kappa\left({s_L}^{\mathrm{T}}s_L\right)^{1/p}-k_v\left(\Psi^{\mathrm{T}}\Psi\right)^{1/p}+k_ps_{L_{max}}\epsilon \tag{74}$$

On the boundary of the neighborhood $\mathcal{N}$ defined by (67), the upper bound on $\dot{\mathcal{V}}$ is given by

$$\dot{\mathcal{V}}\leq-k_v\left(s_{L_{max}}^{(2/p)}+\Psi_{max}^{(2/p)}\right)+k_ps_{L_{max}}\epsilon. \tag{75}$$

Therefore, $\dot{\mathcal{V}}$ is non-positive along the boundary of $\mathcal{N}$ if

$$-k_v\left(s_{L_{max}}^{(2/p)}+\Psi_{max}^{(2/p)}\right)+k_ps_{L_{max}}\epsilon\leq 0, \tag{76}$$

which is a sufficient condition for all trajectories starting outside this neighborhood of $(I, 0, 0)$ to converge to it. Alternately, expression (76) also leads to the following expression for the ratio of observer gains:

$$\frac{k_p}{k_v}\leq\frac{\left(s_{L_{max}}^{(2/p)}+\Psi_{max}^{(2/p)}\right)}{\epsilon\ s_{L_{max}}}. \tag{77}$$

This relates the ratio $\dfrac{k_p}{k_v}$ to the bound on the norm of the noise ν and bounds on the neighborhood $\mathcal{N}$, for which convergence of estimation errors $(Q, \tilde{\Omega}, \tilde{\beta})$ to this neighborhood of $(I, 0, 0)$ is guaranteed. ■

6. Other state-of-the-art filters on SO(3)

Three estimation schemes are used in comparisons with the finite-time stable variational estimator (FTS): The discrete-time variational estimator (VAE), the geometric approximate minimum-energy (GAME) estimator, and a constant gain observer (CGO).

6.1. Discrete-time variational attitude estimator

This estimator appeared in Izadi, Viswanathan, Sanyal, Silvestre, and Oliveira (2016b) and is obtained by applying the (discrete) Lagrange–d'Alembert principle of variational mechanics to a (discrete) Lagrangian constructed from residuals between measurements and state estimates with a dissipation term that is linear in the angular velocity measurement residual. This discrete-time estimator is based on the earlier (continuous-time) variational attitude estimator (VAE) that appeared in Izadi and Sanyal (2014). Here, we generalize the discrete-time VAE to include angular velocity measurements that have a constant bias in addition to measurement noise. The filter equations in discrete-time for a rigid body with the attitude kinematics (25) and with measurements of vectors and angular velocity in a body-fixed frame, are given by

$$\widehat{R}_{k+1}=\widehat{R}_k\ \exp\left(h(\Omega_k^m-\omega_k-\widehat{\beta}_k)^{\times}\right), \tag{78}$$

$$\widehat{\beta}_{k+1}=\widehat{\beta}_k+h\Phi'\left(\mathcal{U}^0(\widehat{R}_k, U_k^m)\right)P^{-1}s_{L_k}(\widehat{R}), \tag{79}$$

$$\widehat{\Omega}_k=\Omega_k^m-\omega_k-\widehat{\beta}_k, \tag{80}$$

$$m\omega_{k+1}=\exp(-h\widehat{\Omega}_{k+1}^{\times})\{(m\ I_{3\times 3}-h\ D)\omega_k \tag{81}$$
$$+h\Phi'\left(\mathcal{U}^0(\widehat{R}_{k+1}, U_{k+1}^m)\right)s_{L_{k+1}}(\widehat{R}_{k+1})\},$$

where $h=t_{k+1}-t_k$ is the time step size for $k=1,2,\ldots,N$, $s_{L_k}(\widehat{R})=\mathrm{vex}({L_k}^{\mathrm{T}}\widehat{R}_k-\widehat{R}_k^{\mathrm{T}}L_k)\in\mathbb{R}^3$, L_k is as defined in (16) and evaluated at time t_k, m is a positive scalar, D is a positive definite filter gain matrix, and $\mathcal{U}^0(\widehat{R}_k, U_k^m)$ and $\Phi\left(\mathcal{U}^0(\widehat{R}, U^m)\right)$ are as defined in Izadi et al. (2016b).

6.2. GAME filter

This estimator is a near-optimal filter proposed in Zamani et al. (2011) by generalizing Mortensen's maximum-likelihood filtering scheme to SO(3). The geometric approximate minimum-energy (GAME) filter in continuous form is as given below:

$$\dot{\widehat{R}}=\widehat{R}(\Omega^m-\widehat{\beta}+P_al)^{\times}, \tag{82}$$

$$\text{where}\ \ l=\sum_{i=1}^{j}\left(\mathscr{D}_i(\widehat{u}_i-u_i)\right)\times\widehat{u}_i$$

$$\dot{P}_a=\mathcal{Q}_\Omega+2\mathbb{P}_s\left(P_a(2(\Omega^m-\widehat{\beta})-P_al)^{\times}\right) \tag{83}$$
$$+P_a(\mathbb{E}-\mathbb{S})P_a-{P_c}^{\mathrm{T}}-P_c,$$

$$\dot{P}_c=-(\Omega^m-\widehat{\beta}-P_al)^{\times}P_c+P_a(\mathbb{E}-\mathbb{S})P_c-P_b, \tag{84}$$

$$\dot{P}_b=\mathcal{Q}_b+P_c(\mathbb{E}-\mathbb{S})P_c, \tag{85}$$

$$\dot{\widehat{\beta}}={P_c}^{\mathrm{T}}\ l, \tag{86}$$

where

$$\mathbb{S}=\sum_{i=1}^{j}(\widehat{u}_i^{\times})^{\mathrm{T}}\mathscr{D}_i\widehat{u}_i^{\times}, \tag{87}$$

$$\mathbb{E}=\mathrm{trace}(C)I-C, \tag{88}$$

$$C=\sum_{i=1}^{j}\mathbb{P}_s\left(\mathscr{D}_i(\widehat{u}_i-u_i){\widehat{u}_i}^{\mathrm{T}}\right). \tag{89}$$

Here u_i is the true vector observed in body frame, $\hat{u}_i = \widehat{R}^{\mathrm{T}} e_i$, $\mathcal{Q}_{\Omega} = BB^{\mathrm{T}}$ where $B \in \mathbb{R}^{3\times 3}$ allows for different weights for the components of the vector of additive noise in angular velocity components, $Q_b = I_{3\times 3}$, $\mathbb{P}_s(X) = \frac{1}{2}(X + X^{\mathrm{T}})$ for $X \in \mathbb{R}^{3\times 3}$, $\mathscr{D}_i = (\mathcal{D}_i \mathcal{D}_i^{\mathrm{T}})^{-1}$ where $\mathcal{D}_i \in \mathbb{R}^{3\times 3}$ allows for different weights for the vector measurement noise, $\widehat{R}(0) = I_{3\times 3}$, $P_c(0) = 0_{3\times 3}$, $P_a(0) = \frac{1}{\varphi^2} I_{3\times 3}$ where φ^2 is the variance of the principal angle corresponding to the initial attitude estimate, and $P_b(0) = \frac{1}{\psi^2} I_{3\times 3}$ where ψ^2 is the standard deviation of an initial bias.

6.3. The Constant Gain Observer

The Constant Gain Observer (CGO) presented in Mahony et al. (2008) in continuous form is also represented as

$$\dot{\widehat{R}} = \widehat{R}\Big(\Omega^m - \widehat{\beta} + K_P \bar{\ell}\Big)^{\times}, \ \bar{\ell} = \sum_{i=1}^{j} (u_i \times \widehat{u}_i), \tag{90}$$

$$\dot{\widehat{\beta}} = K_I \, \ell. \tag{91}$$

where K_P and K_I are constant gains and $\widehat{R}(0) = I_{3\times 3}$. Note that the discrete-time versions of this filter and the GAME filter as presented in Zamani (2013) use the unit quaternion representation, and are implemented as such in the following numerical simulations.

7. Numerical simulation results

This section presents some numerical simulation results of the proposed FTS estimation scheme as well as the results of a comparison between this scheme and three other state-of-the-art estimators. First, the simulation results of the proposed scheme are presented in the following subsection.

7.1. Numerical results of the finite-time stable (FTS) estimator in the absence of noise

In this subsection, simulation results of the proposed finite-time stable (FTS) estimator *without any measurement noise* are presented to show the finite-time convergence of all estimation errors to zero. Attitude estimation with the FTS estimator as well as the discrete VAE introduced in Section 6.1 are numerically implemented using a *geometric scheme*. Unlike commonly used numerical integration methods like Runge–Kutta, geometric integration schemes preserve the geometry of the state space without any projection or parameterization. Let $\dot{\omega} = \chi$ where χ is the right-hand side of (51) divided by μ. Let $h = t_{k+1} - t_k$ be the time step size. Discretized equations that are used to numerically implement the proposed FTS estimation scheme are as follows:

$$\widehat{R}_{i+1} = \widehat{R}_i \exp\!\big(h(\Omega_i^m - \widehat{\beta}_i - \omega_i)^{\times}\big), \tag{92}$$

$$\omega_{i+1} = \omega_i + h\chi_i, \tag{93}$$

$$\widehat{\beta}_{i+1} = \widehat{\beta}_i + \frac{h\,k_p}{2-\mu} s_{L_i}(\widehat{R}_i). \tag{94}$$

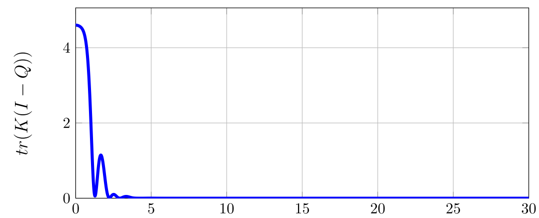
where

$$\chi_i = \mu^{-1}\Bigg(k_p s_{L_i} - \frac{k_v \Psi_i}{\big(\Psi_i{}^{\mathrm{T}}\Psi_i\big)^{1-1/p}} + \frac{\mu\kappa H(s_{L_i}) w_{L_i}}{\big(s_{L_i}{}^{\mathrm{T}} s_{L_i}\big)^{1-1/p}}\Bigg) \tag{95}$$

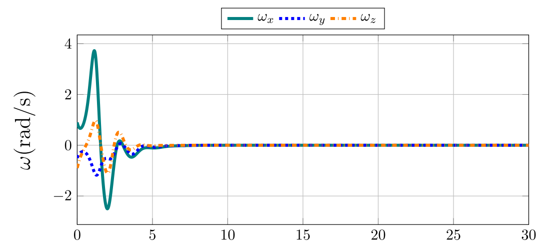
The matrix exponential map in (92) guarantees that each attitude estimate belongs to SO(3).

The estimator is simulated with a time step size of $h = 0.01$ s for a time duration of $T = 30$ s. The rigid body is assumed to have the following initial attitude and angular velocity:

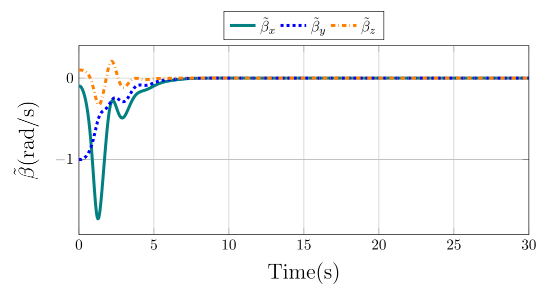
$$R_0 = \exp\Big(\pi\big([1,\ 0,\ 0]^{\mathrm{T}}\big)^{\times}\Big), \quad \Omega_0 = [1\ 0.5\ 0]^{\mathrm{T}}\mathrm{rad/s}.$$



(a) Attitude Estimation Error with time



(b) Angular velocity error ω with time



(c) Bias estimation error $\tilde{\beta}$ with time

Fig. 1. Simulation results.

The initial estimated states are selected to be $\widehat{R}_0 = I$, $\widehat{\Omega}_0 = [0,\ 0,\ 1]^{\mathrm{T}}$rad/s, and $\widehat{\beta}_0 = [0,\ 0,\ 0.1]^{\mathrm{T}}$. Three inertial vectors are considered to be measured at a constant rate by body-fixed sensors. There is no measurement noise in the direction vector measurements or angular velocity vector measurement. The angular velocity measurement is only assumed to have a constant bias of $\beta = [-0.1,\ -1,\ 0.2]^{\mathrm{T}}$ rad/s. The observer gains are $k_p = 2$, $k_v = 1$, $\kappa = 0.1$, and $\mu = 0.35$. The fractional exponent is taken as $p = 1.1$.

The simulation results are illustrated in Fig. 1. The attitude error, Q, error in estimation of angular velocity and bias, ω, and bias estimation error itself, $\tilde{\beta}$, are shown to converge in finite time in the absence of measurement noise, which implies the finite time stability of the estimation scheme.

7.2. Comparison results of the FTS attitude estimator with other attitude estimators in the presence of noise

In this subsection, the performance of the finite-time stable estimator *in the presence of vector measurement noise, σ_i, and angular velocity noise, ν_i, and unknown bias in angular velocity measurements* is compared to that of the three estimators presented in Section 6, under identical conditions. To do so, all the estimation schemes are applied to the same rigid-body dynamics, with the same initial estimate errors, equal time steps, and identical measurement noise. The sampling period and the total simulation time are $h = 0.01$ s and $T = 30$ s, respectively. Three known inertial directions are measured by the sensors in body frame, and these measurements include known levels of noise, and rate

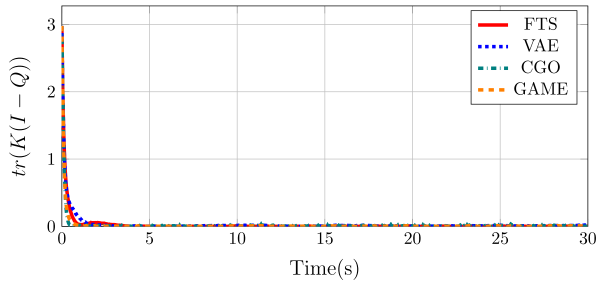


Fig. 2. Attitude estimation error for noise levels similar to that in Zamani (2013).

gyro sensors for angular velocity measurement are assumed to be biased with a constant bias of $\beta = [-0.01,\ -0.005,\ 0.02]^{\mathrm{T}}$ rad/s for all schemes. The initial estimate of the bias is set to be $\widehat{\beta}_0 = [0,\ 0,\ 0.1]^{\mathrm{T}}$, the initial estimated rotation matrix is set equal to identity, and the initial rotation matrix is selected such that its principal angle has zero mean and a standard deviation of 60°. The rigid body simulated has the following angular velocity profile:

$$\Omega(t) = \left[\sin\frac{2\pi}{15}t \quad -\sin\frac{2\pi}{18}(t+\frac{\pi}{20}) \quad \cos(\frac{2\pi}{17}t)\right]^{\mathrm{T}} \tag{96}$$

The initial angular velocity estimates are also set to be identical, as follows. According to (82), the initial angular velocity estimation error is given by $P_a(0)l(0)$ for the GAME filter. For the FTS and discrete-time VAE estimators, choosing $\omega_0 = P_a(0)l(0)$ and for the CGO, choosing $K_P = P_a(0)$ satisfies this condition. For this comparison in simulations, the scalar "inertia-like" gain for the VAE estimator is selected as $m = 0.5$, the constant gain K_I of CGO filter is set equal to 0.3, and the positive definite dissipation matrix as

$$D = \mathrm{diag}\Big(\big[2.4\ \ 2.6\ \ 2.8\big]^{\mathrm{T}}\Big). \tag{97}$$

As in Zamani (2013), the GAME and CGO estimators utilize unit quaternions for attitude representation when implemented numerically. We compare the performance of these two estimators as well as the discrete VAE scheme with the FTS attitude estimator for two different cases, as described in the rest of this section.

7.2.1. CASE I: High noise levels

In this case, both direction vector measurement noise vectors σ_i and angular velocity measurement noise vector ν are random zero mean signals whose probability distributions are normalized bump functions. It is assumed that the standard deviation of the direction measurement noise and angular velocity measurement noise are 30° and 25°/s, respectively. In order to have a fair comparison between the different estimation schemes that may have different gains (and gain update scheme in the case of the GAME filter), the (initial) gains are selected such that all estimators have the same initial attitude and initial angular velocity estimates. Moreover, all estimators simulated here are provided the same set of measurements with the same (constant) bias added to the angular velocity measurements. The time profiles of the attitude estimate error for each estimator are plotted and compared in Fig. 2.

Fig. 2 shows some transient behavior in the attitude estimation error with the FTS estimator. However, there are no remarkable differences in the steady state behavior of all four schemes compared, and in fact the constant gain observer performs somewhat better than the other schemes. The FTS estimator shows finite-time convergence of attitude estimate error to zero, and the settling time for this estimator is comparable to that of the other filters.

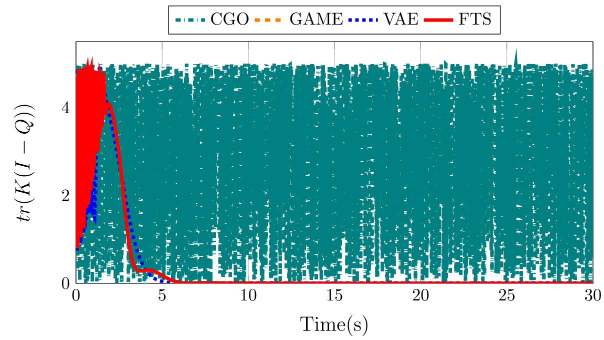


Fig. 3. Attitude estimation error for low noise levels, with estimator gains unchanged.

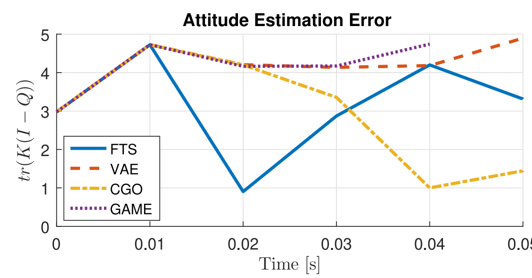


Fig. 4. Zoomed-in view of the initial transient response of the attitude estimation error as plotted in Fig. 3.

7.2.2. CASE II: Low noise levels, with estimator gains as before

In this case, the noise signals are considered to be of the same type as in the previous case (random zero mean bump functions), but with much smaller amplitudes. The standard deviations in the attitude measurement noise and angular velocity measurement noise signals are 0.95° and 2.5°/s, respectively. This corresponds to having more accurate sensor measurements than in case I. In order to compare the estimator performances when the estimator gains are not designed for known sensor noise properties, all the gains are kept the same as in case I. The attitude estimation errors from all estimators are plotted in Fig. 3.

A magnified view of the initial transient behavior of these observers is depicted in Fig. 4. In this case, as is shown in Fig. 4, the GAME filter becomes singular after a few time steps, and the CGO is not able to converge and filter out noise from the measurements. On the other hand, the FTS and VAE estimators are stable and very effective at filtering out noise. Moreover, the FTS estimator guarantees convergence of estimation errors to small bounds in finite-time. The settling times are also sufficiently small. Looking at these two cases, one can conclude that although the GAME and CGO filters perform nicely in the presence of measurement noise with known noise level, they may not be stable and their initial gains need to be reset when the noise signal changes. Therefore, one major downside to these filters is their dependence on the knowledge of the measurement noise level. On the contrary, the variational estimators (FTS and VAE) are robust with guaranteed stability, regardless of the statistics and level of the noise. Moreover, because of the almost global finite-time stable property of the FTS estimator, it is robust to bounded measurement noise in attitude states, as shown in this paper.

8. Conclusion and future work

This paper presents a nonlinear finite-time stable state estimator for rigid body rotational motion. The proposed scheme estimates the attitude and constant angular velocity bias vector from a minimum of two known linearly independent vectors for attitude, and biased angular velocity measurements made in the body-fixed frame. The estimation errors including the bias estimation error are analytically proven to stabilize to zero from almost all initial conditions in the absence of measurement errors. The scheme is numerically implemented by a geometric integrator for a realistic scenario involving measurement errors. Numerical results validate the theoretical results and show the robustness of the proposed estimation scheme. The behavior of this estimation scheme is compared with three state-of-the-art filters for attitude estimation. Using a realistic set of data for a rigid body, numerical simulations show that the FTS and variational attitude estimator (VAE), unlike the GAME filter and CGO, are always stable and their convergence is not dependent on the type and level of measurement noise. Moreover, finite-time stability guarantees a faster convergence of estimation errors $(Q, \tilde{\Omega}, \tilde{\beta})$ to $(I, 0, 0)$ in finite-time, and robustness to measurement noise. Future research shall look into the problem of discrete-time stable attitude estimation from intermittent measurements.

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