4 Stable Estimation of Rigid Body Motion Based on the Lagrange–d'Alembert Principle

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CONTENTS

ABSTRACT Stable estimation of rigid body rotational and translational motion states from noisy measurements, without any knowledge of the dynamics model, is treated using the Lagrange–d'Alembert principle from variational mechanics. With body-fixed sensor measurements, a Lagrangian is obtained as the difference between a kinetic energy-like term that is quadratic in velocity estimation errors and an artificial potential function of configuration (attitude and position) estimation errors. An additional dissipation term that is linear in the velocity estimation errors is introduced, and the Lagrange–d'Alembert principle is applied to the Lagrangian with this dissipation. This estimation scheme is shown to be almost globally asymptotically stable in the state space of rigid body motions. It is discretized for computer implementation using the discrete Lagrange–d'Alembert principle, as a first-order Lie group variational integrator (LGVI). In the presence of bounded measurement noise in the measurements, numerical simulations show that the estimated states converge to a bounded neighborhood of the actual states.

4.1 INTRODUCTION

Estimation of rigid body translational and rotational motion is indispensable for operations of spacecraft, unmanned aerial, and underwater vehicles. Autonomous state estimation of a rigid body based on inertial vector measurement and visual feedback from stationary landmarks, in the absence of a dynamics model for the rigid body, is analyzed here. This estimation scheme can enhance the autonomy and reliability of unmanned vehicles in uncertain GPS-denied environments. In practice, the dynamics of a vehicle may not be perfectly known, especially when the vehicle is under the action of poorly known forces and moments.

Attitude estimators using unit quaternions for attitude representation may be *unstable in the sense of Lyapunov*, unless they identify antipodal quaternions with a single attitude. This is also the case for attitude control schemes based on continuous feedback of unit quaternions. One adverse consequence of these unstable estimation and control schemes is that they end up taking longer to converge compared to stable schemes under similar initial conditions and initial transient behavior. Continuous-time attitude observers and filtering schemes on SO(3) and SE(3) do not suffer from kinematic singularities like estimators using coordinate descriptions of attitude, and they do not suffer from unwinding as they do not use unit quaternions. The maximum-likelihood (minimum energy) filtering method of Mortensen [1] was recently applied to attitude estimation, resulting in a nonlinear attitude estimation scheme that seeks to minimize the stored *energy* in measurement errors [2]. This scheme is obtained by applying Hamilton–Jacobi–Bellman (HJB) theory to the state space of attitude motion. Since the HJB equation can only be approximately solved with increasingly unwieldy expressions for higher order approximations, the resulting filter is only *near optimal* up to second order. Unlike filtering schemes that are based on approximate or *near optimal* solutions of the HJB equation and do not have provable stability, the estimation scheme obtained here can be solved exactly. Moreover, unlike filters based on Kalman filtering, the estimators proposed here do not presume any knowledge of the statistics of the initial state estimate or the sensor noise.

The variational attitude and pose estimation schemes recently appeared in [3,4], where they were shown to be almost globally asymptotically stable. The framework of variational estimation overcomes some of the issues encountered by competing schemes as outlined in the previous two paragraphs; these advantages were reported in [5]. This chapter outlines the variational estimation approach for two different cases: (1) rotational (attitude) motion only, which is on the *special orthogonal* group SO(3) and (2) coupled rotational and translational motion, which is on the *special Euclidean* group SE(3).

4.2 ATTITUDE AND ANGULAR VELOCITY ESTIMATION USING VECTOR AND ANGULAR VELOCITY MEASUREMENTS

Rigid body attitude is determined from $j \in \mathbb{N}$ known inertial vectors measured in a coordinate frame fixed to the rigid body. Let these vectors be denoted as l_j^m for $j = 1, 2, \ldots, j$, in the body-fixed frame. The assumption that $j \geq 2$ is necessary for instantaneous three-dimensional attitude determination. When $j = 2$, the cross product of the two measured vectors is considered as a third measurement for applying the attitude estimation scheme. Denote the corresponding known inertial vectors as seen from the rigid body as d_j, and let the true vectors in the body frame be denoted $l_j = R^{\mathrm{T}} d_j$, where R is the rotation matrix from the body frame to the inertial frame. This rotation matrix provides a coordinate-free, global, and unique description of the attitude of the rigid body. Define the matrix composed of all j measured vectors expressed in the body-fixed frame as column vectors,

$$L^m = [l_1^m \; l_2^m \; l_1^m \times l_2^m] \text{ when } j = 2, \text{ and } L^m = [l_1^m \; l_2^m \ldots l_j^m] \text{ when } j > 2, \tag{4.1}$$

and the corresponding matrix of all these vectors expressed in the inertial frame as

$$D = [d_1\ d_2\ d_1 \times d_2] \text{ when } j = 2, \text{ and } D = [d_1\ d_2 \ldots d_j] \text{ when } j > 2. \tag{4.2}$$

The matrix of true body vectors l_j corresponding to the inertial vectors d_j, is given by

$$L = R^{\mathrm{T}}D = [l_1\ l_2\ l_1 \times l_2] \text{ when } j = 2, \text{ and } L = R^{\mathrm{T}}D = [l_1\ l_2 \ldots l_j] \text{ when } j > 2. \tag{4.3}$$

4.2.1 Generalization of Wahba's Cost Function

The optimal attitude determination problem for a set of vector measurements at a given time instant, is to find an estimated rotation matrix $\hat{R} \in \mathrm{SO}(3)$ such that a weighted sum of the squared norms of the vector errors

$$s_j = d_j - \hat{R} l_j^m \tag{4.4}$$

is minimized. This attitude determination problem is known as *Wahba's problem*, and it is the problem of minimizing the value, with respect to $\hat{R} \in \mathrm{SO}(3)$, of

$$\mathcal{U}_r^0(\hat{R}, L^m) = \frac{1}{2}\sum_{j=1}^{n} w_j \left(d_j - \hat{R}\, l_j^m \right)^{\mathrm{T}} \left(d_j - \hat{R}\, l_j^m \right), \tag{4.5}$$

where the weights $w_j > 0$, $n = 3$ if $j = 2$ and $n = j$ if $j \geq 2$. Defining the trace inner product on $\mathbb{R}^{n_1 \times n_2}$ as

$$\langle A_1, A_2 \rangle = \mathrm{trace}(A_1^{\mathrm{T}} A_2), \tag{4.6}$$

we can re-express Equation 4.5 for Wahba's cost function as

$$\mathcal{U}_r^0(\hat{R}, L^m) = \frac{1}{2}\left\langle D - \hat{R}L^m,\ (D - \hat{R}L^m)W \right\rangle, \tag{4.7}$$

where:

- L^m is given by Equation 4.1
- D is given by Equation 4.2
- $W = \mathrm{diag}(w_j)$ is the positive diagonal matrix of the weight factors for the measured directions

From the above expression (Equation 4.7), note that W can be any positive definite matrix, not necessarily diagonal. Another generalization of Wahba's cost function is given by

$$\mathcal{U}_r(\hat{R}, L^m) = \Phi\left(\frac{1}{2}\left\langle D - \hat{R}L^m,\ (D - \hat{R}L^m)W \right\rangle \right), \tag{4.8}$$

where Φ: $[0,\infty) \mapsto [0, \infty)$ is a $\mathcal{C}^2$ function that satisfies $\Phi(0) = 0$ and $\Phi'(\mathcal{X}) > 0$ for all $\mathcal{X} \in [0, \infty)$. Furthermore, $\Phi'(\cdot) \leq \alpha(\cdot)$ where $\alpha(\cdot)$ is a class-$\mathcal{K}$ function [6] and $\Phi'(\cdot)$ denotes the derivative of $\Phi(\cdot)$ with respect to its argument. Note that these properties of $\Phi(\cdot)$ ensure that the indices $\mathcal{U}_r^0(\hat{R}, L^m)$ and $\mathcal{U}_r(\hat{R}, L^m)$ have the same minimizer $\hat{R} \in \mathrm{SO}(3)$. In other words, minimizing the cost $\mathcal{U}_r$, which is a generalization of the cost $\mathcal{U}_r^0$, is equivalent to solving Wahba's problem. Here, W is positive definite (not necessarily diagonal), and D and L^m are of rank 3 when $j \geq 2$ vectors are measured.

4.2.2 Choice of Weights for Wahba's Cost Function

In the absence of measurement errors, $L^m = L = R^{\mathrm{T}}D$, and let $Q = R\hat{R}^{\mathrm{T}} \in \mathrm{SO}(3)$ denote the attitude estimation error. The following lemma specifies the weight matrix W according to the SVD of D and selected eigenvalues ς_1, ς_2, $\varsigma_3 > 0$ for the matrix $K = DWD^{\mathrm{T}}$.

Lemma 4.1

*Let rank (*D*) = 3. Let the singular value decomposition of* D *be given by*

$$D := U_D \Sigma_D V_D^T \quad \text{where} \quad U_D \in \mathrm{O}(3), \quad V_D \in \mathrm{O}(n), \Sigma_D \in \mathrm{Diag}^+(3,n), \tag{4.9}$$

and $\mathrm{Diag}^+(n_1, n_2)$ *is the vector space of* $n_1 \times n_2$ *matrices with positive entries along the main diagonal and all other components zero. Let* σ_1, σ_2, σ_3 *denote the main diagonal entries of* Σ_D. *Furthermore, let the positive definite weight matrix W be given by*

$$W = V_D W_0 V_D^T \quad \text{where} \quad W_0 \in \mathrm{Diag}^+(n,n) \tag{4.10}$$

and the first three diagonal entries of W_0 *are given by*

$$w_1 = \frac{\varsigma_1}{\sigma_1^2}, \quad w_2 = \frac{\varsigma_2}{\sigma_2^2}, \quad w_3 = \frac{\varsigma_3}{\sigma_3^2} \quad \text{where} \quad \varsigma_1, \varsigma_2, \varsigma_3 > 0. \tag{4.11}$$

Then, $K = DWD^T$ *is positive definite and*

$$K = U_D \Delta U_D^T \quad \text{where} \quad \Delta = \mathrm{diag}(\varsigma_1, \varsigma_2, \varsigma_3), \tag{4.12}$$

is its eigendecomposition. Moreover, if $\varsigma_i \neq \varsigma_j$ *for* $i \neq j$ *and* $i, j \in \{1, 2, 3\}$, *then* $\langle I - Q, K\rangle$ *is a Morse function.*

4.2.3 Attitude Kinematics

Let $\Omega \in \mathbb{R}^3$ be the angular velocity of the rigid body expressed in the body-fixed frame, and $\hat{\Omega}$ denote its estimate. The attitude kinematics and estimated attitude kinematics, respectively, are given by

$$\dot{R} = R\Omega^\times, \quad \dot{\hat{R}} = \hat{R}\hat{\Omega}^\times, \tag{4.13}$$

where $(\cdot)^\times : \mathbb{R}^3 \to \mathfrak{so}(3) \subset \mathbb{R}^{3\times 3}$ is the skew-symmetric cross-product operator. In order to obtain attitude state estimation schemes from continuous-time vector and angular velocity measurements, we apply the Lagrange–d'Alembert principle to an action functional of a Lagrangian of the state estimate errors, with a dissipation term in the angular velocity estimate error. Section 4.2 presents an estimation scheme obtained by using this approach.

4.2.4 Action Functional of the Lagrangian

The *energy* contained in the errors between the estimated and the measured inertial vectors is given by $\mathcal{U}_r(\hat{R}, L^m)$, where $\mathcal{U}_r : \mathrm{SO}(3) \times \mathbb{R}^{3\times n} \to \mathbb{R}$ is defined by Equation 4.8 and depends on the attitude estimate. The *energy* contained in the vector error between the estimated and the measured angular velocity is given by

$$\mathcal{T}(\hat{\Omega}, \Omega^m) = \frac{m}{2}(\Omega^m - \hat{\Omega})^T(\Omega^m - \hat{\Omega}). \tag{4.14}$$

where m is a positive scalar. One can consider the Lagrangian composed of these *energy* quantities as follows:

$$\begin{aligned}\mathcal{L}(\hat{R}, L^m, \hat{\Omega}, \Omega^m) &= \mathcal{T}(\hat{\Omega}, \Omega^m) - \mathcal{U}_r(\hat{R}, L^m) \\ &= \frac{m}{2}(\Omega^m - \hat{\Omega})^T(\Omega^m - \hat{\Omega}) - \Phi\left(\frac{1}{2}\left\langle D - \hat{R}L^m,\ (D - \hat{R}L^m)W \right\rangle\right).\end{aligned} \tag{4.15}$$

If the estimation process is started at time t_0, then the action functional of the Lagrangian (Equation 4.15) over the time duration $[t_0, T]$ is expressed as

$$\begin{aligned}\mathcal{S}(\mathcal{L}(\hat{R}, L^m, \hat{\Omega}, \Omega^m)) &= \int_{t_0}^{T}\left(\mathcal{T}(\hat{\Omega}, \Omega^m) - \mathcal{U}_r(\hat{R}, L^m)\right)\mathrm{d}t \\ &= \int_{t_0}^{T}\left\{\frac{m}{2}(\Omega^m - \hat{\Omega})^T(\Omega^m - \hat{\Omega}) - \Phi\left(\frac{1}{2}\left\langle D - \hat{R}L^m,\ (D - \hat{R}L^m)W \right\rangle\right)\right\}\mathrm{d}t.\end{aligned} \tag{4.16}$$

4.2.5 Variational Estimator for Attitude and Angular Velocity

Consider attitude state estimation in continuous time in the presence of measurement noise and initial-state estimate errors. Applying the Lagrange–d'Alembert principle to the action functional $\mathcal{S}(\mathcal{L}(\hat{R}, L^m, \hat{\Omega}, \Omega^m))$ given by Equation 4.16, in the presence of a dissipation term on $\omega := \Omega^m - \hat{\Omega}$, leads to the following attitude and angular velocity filtering scheme. This estimation scheme was shown to be almost globally asymptotically stable in [3].

Theorem 4.1

The filter equations for a rigid body with the attitude kinematics (Equation 4.13) and with measurements of vectors and angular velocity in a body-fixed frame are of the form

$$\begin{cases}\dot{\hat{R}} = \hat{R}\hat{\Omega}^\times = \hat{R}(\Omega^m - \omega)^\times, \\ m\dot{\omega} = -m\hat{\Omega}\times\omega + \Phi'\left(\mathcal{U}_r^0(\hat{R}, L^m)\right)\mathcal{S}_\Gamma(\hat{R}) - \mathcal{D}\omega, \\ \hat{\Omega} = \Omega^m - \omega,\end{cases} \tag{4.17}$$

where $\mathcal{D}$ is a positive definite filter gain matrix, $\hat{R}(t_0) = \hat{R}_0$, $\omega(t_0) = \omega_0 = \Omega_0^m - \hat{\Omega}_0$, $\mathcal{S}_\Gamma(\hat{R}) = \mathrm{vex}\left(\Gamma^T\hat{R} - \hat{R}^T\Gamma\right) \in \mathbb{R}^3$, $\Gamma = DW(L^m)^T$, $\mathrm{vex}(\cdot): \mathfrak{so}(3) \to \mathbb{R}^3$ is the inverse of the $(\cdot)^\times$ map and W is chosen to satisfy the conditions in Lemma 4.1.

Proof. In order to find a filter equation that reduces the measurement noise in the estimated attitude, one may take the first variation of the action functional (Equation 4.16) with respect to $\hat{R}$ and $\hat{\Omega}$. Consider the potential term $\mathcal{U}_r^0(\hat{R}, L^m)$ as defined by Equation 4.7. Taking the first variation of this function with respect to $\hat{R}$ gives

$$\begin{aligned}\delta\mathcal{U}_r^0 &= \left\langle -\delta\hat{R}L^m, (D - \hat{R}L^m)W \right\rangle = \frac{1}{2}\left\langle \Sigma^\times, L^m W D^T\hat{R} - \hat{R}^T DW(L^m)^T \right\rangle, \\ &= \frac{1}{2}\left\langle \Sigma^\times, \Gamma^T\hat{R} - \hat{R}^T\Gamma \right\rangle = \mathcal{S}_\Gamma^T(\hat{R})\Sigma.\end{aligned} \tag{4.18}$$

Now consider $\mathcal{U}_r(\hat{R},L^m) = \Phi\left(\mathcal{U}_r^0(\hat{R},L^m)\right)$. Then,

$$\delta\mathcal{U}_r = \Phi'\left(\mathcal{U}_r^0(\hat{R},L^m)\right)\delta\mathcal{U}_r^0 = \Phi'\left(\mathcal{U}_r^0(\hat{R},L^m)\right)\mathcal{S}_\Gamma^{\mathrm{T}}(\hat{R})\Sigma. \tag{4.19}$$

Taking the first variation of the kinematic energy term associated with the artificial system (Equation 4.14) with respect to $\hat{\Omega}$ yields

$$\delta\mathcal{T} = -m(\Omega^m - \hat{\Omega})^{\mathrm{T}}\delta\hat{\Omega} = -m(\Omega^m - \hat{\Omega})^{\mathrm{T}}(\dot{\Sigma} + \hat{\Omega}\times\Sigma) = -m\omega^{\mathrm{T}}(\dot{\Sigma} + \hat{\Omega}\times\Sigma), \tag{4.20}$$

where $\omega = \Omega^m - \hat{\Omega}$. Applying the Lagrange–d'Alembert principle leads to

$$\begin{aligned}\delta\mathcal{S} + \int_{t_0}^{T}\tau_{\mathcal{D}}^T\Sigma \mathrm{d}t = 0 &\Rightarrow \int_{t_0}^{T}\left\{-m\omega^T(\dot{\Sigma} + \hat{\Omega}\times\Sigma) - \Phi'\left(\mathcal{U}_r^0(\hat{R},L^m)\right)\mathcal{S}_\Gamma^T(\hat{R})\Sigma + \tau_{\mathcal{D}}^T\Sigma\right\}\mathrm{d}t = 0\\ &\Rightarrow -m\omega^T\Sigma\Big|_{t_0}^{T} + \int_{t_0}^{T} m\dot{\omega}^T\Sigma\mathrm{d}t = \int_{t_0}^{T}\left\{m\omega^T\hat{\Omega}^\times + \Phi'\left(\mathcal{U}_r^0(\hat{R},L^m)\right)\mathcal{S}_\Gamma^T(\hat{R}) - \tau_{\mathcal{D}}^T\right\}\Sigma\mathrm{d}t,\end{aligned} \tag{4.21}$$

where the first term in the left-hand side vanishes, since $\Sigma(t_0) = \Sigma(T) = 0$, and after replacing the dissipation term $\tau_{\mathcal{D}} = \mathcal{D}\omega$ gives the second equation in Equation 4.17.

4.2.6 Discrete-Time Variational Attitude Estimator

For onboard computer implementation, the variational estimation scheme outlined earlier has to be discretized. This discretization is carried out in the framework of discrete geometric mechanics, and the resulting discrete-time estimator is in the form of an LGVI. A variational integrator works by discretizing the (continuous-time) variational mechanics principle that leads to the equations of motion, rather than discretizing the equations of motion directly. LGVIs are variational integrators for mechanical systems whose configuration spaces are Lie groups, such as rigid-body systems. In addition to maintaining properties arising from the variational principles of mechanics, such as energy and momenta, LGVI schemes also maintain the geometry of the Lie group, that is, the configuration space of the system.

Consider an interval of time $[t_0,T]\in\mathbb{R}^+$ separated into N equal-length subintervals $[t_i, t_{i+1}]$ for $i = 0, 1, \ldots, N$, with $t_N = T$ and $t_{i+1} - t_i = \Delta t$ is the time step size. Let $(\hat{R}_i,\hat{\Omega}_i)\in \mathrm{SO}(3)\times\mathbb{R}^3$ denote the discrete state estimate at time t_i, such that $(\hat{R}_i,\hat{\Omega}_i)\approx(\hat{R}(t_i),\hat{\Omega}(t_i))$ where $(\hat{R}(t),\hat{\Omega}(t))$ is the exact solution of the continuous-time filter at time $t\in[t_0, T]$, $D_i\in\mathbb{R}^{3\times n}$ is the set of inertial vectors and $L_i^m\in\mathbb{R}^{3\times n}$ is the corresponding set of measured body vectors observed at time t_i, and W_i is the corresponding diagonal matrix of weight factors. It is assumed that these measurements are obtained in discrete-time at a sufficiently high but constant sample rate. The weights in W_i can be chosen such that K_i is always positive definite with distinct (perhaps constant) eigenvalues, as in the continuous-time filter given by Theorem 4.1. The following statement gives the discrete-time filter equations, in the form of a LGVI, corresponding to the continuous-time filter given by Theorem 4.1.

Theorem 4.2

Let two or more vector measurements be available, along with angular velocity measurements in discrete-time, at time intervals of length Δt. Furthermore, let the weight matrix W_i for the set of vector measurements D_i be chosen such that $K_i = D_iW_iD_i^T$ satisfies the eigendecomposition *condition* (*Equation* 4.12) *of Lemma 4.1. A discrete-time filter that approximates the continuous-time filter of Theorem 4.1 to first order in* Δt *is*

$$\begin{cases} \hat{R}_{i+1} = \hat{R}_i \exp(\Delta t \hat{\Omega}_i^{\times}) = \hat{R}_i \exp\left(\Delta t (\Omega_i^m - \omega_i)^{\times}\right), \\ m\omega_{i+1} = \exp(-\Delta t \hat{\Omega}_{i+1}^{\times})\left\{(mI_{3\times 3} - \Delta t \mathcal{D})\omega_i + \Delta t \Phi'\left(\mathcal{U}_r^0(\hat{R}_{i+1}, L_{i+1}^m)\right)\mathcal{S}_{\Gamma_{i+1}}(\hat{R}_{i+1})\right\}, \\ \hat{\Omega}_i = \Omega_i^m - \omega_i, \end{cases} \tag{4.22}$$

where $\mathcal{S}_{\Gamma_i}(\hat{R}_i) = \mathrm{vex}(\Gamma_i^T \hat{R}_i - \hat{R}_i^T \Gamma_i) \in \mathbb{R}^3$, $\Gamma_i = D_i W_i (L_i^m)^T \in \mathbb{R}^{3\times 3}$, *and* $(\hat{R}_0, \hat{\Omega}_0) \in \mathrm{SO}(3) \times \mathbb{R}^3$ *are initial estimated states.*

4.2.7 Numerical Simulations

This section presents numerical simulation results of the discrete time estimator presented in Section 2.6, which is a first-order LGVI. The estimator is simulated over a time interval of $T = 300$ s, with a time stepsize of $\Delta t = 0.01$ s. The rigid body is assumed to have an initial attitude and angular velocity given by

$$R_0 = \mathrm{expm}_{\mathrm{SO(3)}}\left(\left[\frac{\pi}{4} \times \left(\frac{3}{7}\ \frac{6}{7}\ \frac{2}{7}\right)^{\mathrm{T}}\right]^{\times}\right), \quad \text{and} \quad \Omega_0 = \frac{\pi}{60} \times [-2.1\,1.2\,-1.1]^{\mathrm{T}} \text{ rad/s.}$$

The inertia scalar gain is $m = 100$ and the dissipation matrix is selected as the following positive definite matrix:

$$\mathcal{D} = \mathrm{diag}\left([12 \quad 13 \quad 14]^{\mathrm{T}}\right).$$

$\Phi(\cdot)$ could be any C^2 function with the properties described in Section 2.1, but is selected to be $\Phi(\mathcal{X}) = \mathcal{X}$ here. W is selected based on the measured set of vectors D at each instant, such that it satisfies the conditions in Lemma 4.1. The initial estimated states have the following initial estimation errors:

$$Q_0 = \mathrm{expm}_{\mathrm{SO(3)}}\left(\left[\frac{\pi}{2.5} \times \left(\frac{3}{7}\ \frac{6}{7}\ \frac{2}{7}\right)^{\mathrm{T}}\right]^{\times}\right), \text{ and } \omega_0 = [0.001 \quad 0.002 \quad -0.003]^{\mathrm{T}} \text{ rad/s.} \tag{4.23}$$

We assume that there are at most nine inertially known directions that are being measured by the sensors fixed to the rigid body at a constant sample rate. The number of observed directions is taken to be variable over different time intervals. The dynamics equations produce the true states of the rigid body, assuming a sinusoidal force is applied to it. These true states are used to simulate the observed directions in the body-fixed frame, as well as the comparison between true and estimated states. Bounded zero mean noises are considered to be added to the true quantities to generate each measured component. A summation of three sinusoidal matrix functions is added to the matrix $L = R^{\mathrm{T}}D$, to generate a measured L^m with measurement noise. The frequency of the noise signals are 1, 10, and 100 Hz, with different phases and amplitudes up to 2.4°, based on coarse attitude sensors such as sun sensors and magnetometers. Similarly, two sinusoidal noise signals of 10 Hz and 200 Hz frequencies are added to Ω to form the measured Ω^m. These signals also have different phases and their magnitude is up to $0.97/s$, which is close to the real noise levels for coarse rate gyros. In order to integrate the implicit set of equations in Equation 4.22 numerically, the first equation is solved at each sampling step, then the result for R_{i+1} is substituted in the second one. Using the Newton–Raphson method, the resulting equation is solved with respect to ω_{i+1} iteratively. The root of this nonlinear equation with a specific accuracy along with the $\hat{R}_{i+1}$ is used for the next sampling time instant. This process is repeated to the end of the simulation time. Using the aforementioned quantities and the integration method, the simulation is carried out. The principal angle ϕ corresponding to the rigid body's attitude estimation error Q is depicted in Figure 4.1. Components of the estimation error ω in the rigid body's angular velocity are shown in Figure 4.2. All the estimation errors are seen

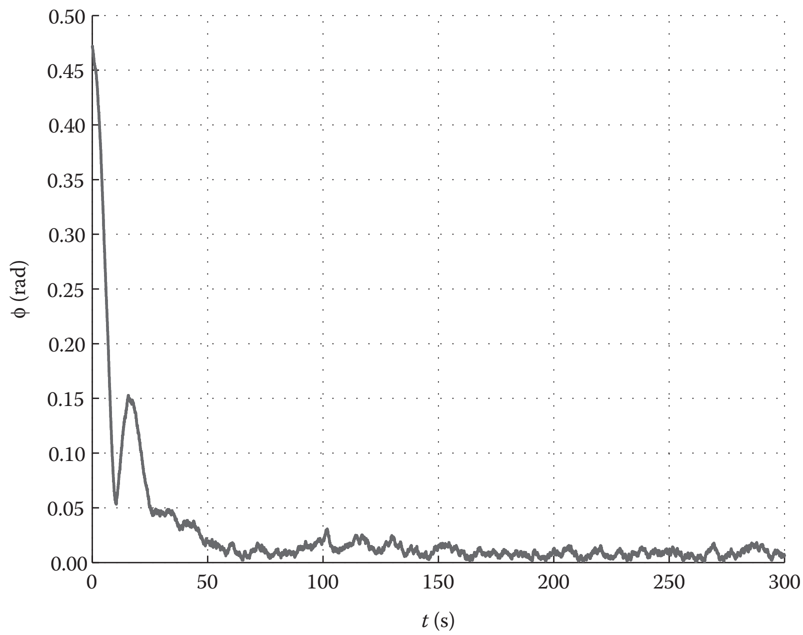

FIGURE 4.1 Principal angle of the attitude estimation error.

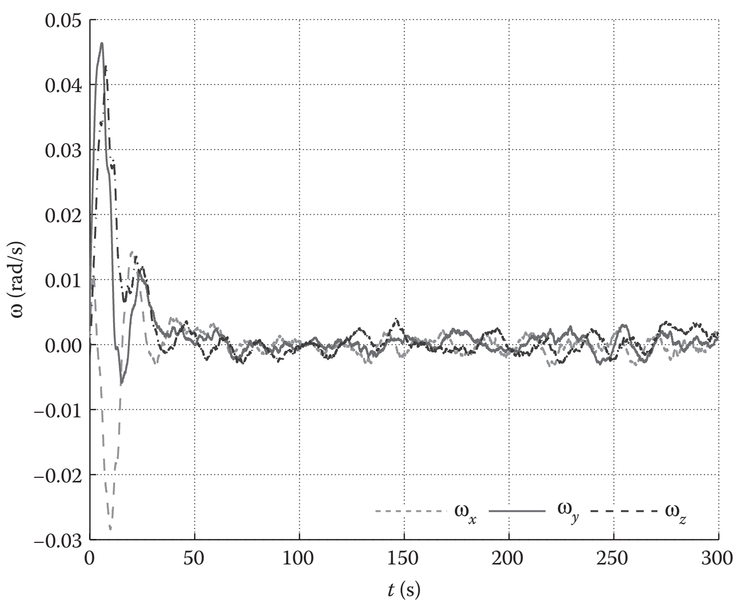

FIGURE 4.2 Angular velocity estimation error.

to converge to a neighborhood of $(Q, \omega) = (I, 0)$, where the size of this neighborhood depends on the bounds of the measurement noise.

4.3 POSE AND VELOCITIES ESTIMATION USING OPTICAL AND INERTIAL SENSORS

Consider a vehicle in spatial (rotational and translational) motion. Onboard estimation of the pose of the vehicle involves assigning a coordinate frame fixed to the vehicle body, and another coordinate frame fixed in the environment that takes the role of the inertial frame. Let O denote the observed environment and S denote the vehicle. Let S denote a coordinate frame fixed to S and O be a coordinate frame fixed to O, as shown in Figure 4.3. Let $R \in \mathrm{SO}(3)$ denote the rotation matrix from frame S to frame O and b denote the position of origin of S expressed in frame O. The pose (transformation) from body fixed frame S to inertial frame O is then given by

$$\mathfrak{g} = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3). \tag{4.24}$$

Consider vectors known in inertial frame O measured by inertial sensors in the vehicle-fixed frame S; let β be the number of such vectors. In addition, consider position vectors of a few stationary points in the inertial frame O measured by optical (vision or lidar) sensors in the vehicle-fixed frame S. Velocities of the vehicle may be directly measured or can be estimated by linear filtering of the optical position vector measurements. Assume that these optical measurements are available for j points at time t, whose positions are known in frame O as p_j, $j \in \mathcal{I}(t)$, where $\mathcal{I}(t)$ denotes the index set of beacons observed at time t. Note that the observed stationary beacons or landmarks may vary over time due to the vehicle's motion. These points generate $\binom{j}{2}$ unique relative position vectors, which are the vectors connecting any two of these landmarks. When two or more position vectors are optically measured, the number of vector measurements that can be used to estimate attitude is $\binom{j}{2} + \beta$. This number needs to be at least two (i.e., $\binom{j}{2} + \beta \geq 2$) at an instant, for the attitude to be uniquely determined at that instant. In other words, if at least two inertial vectors are measured at all instants (i.e., $\beta \geq 2$), then beacon position measurements are not required for estimating attitude. However, at least one beacon or feature point position measurement is still required to estimate the position of the vehicle.

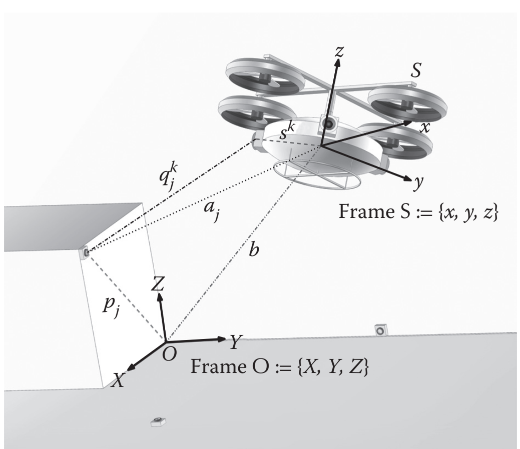

FIGURE 4.3 Inertial landmarks on O as observed from vehicle S with optical measurements.

4.3.1 Pose Measurement Model

Denote the position of an optical sensor and the unit vector from that sensor to an observed beacon in frame S as $s^k \in \mathbb{R}^3$ and $u^k \in \mathbb{S}^2$, $k = 1, \ldots, \not{k}$, respectively. Denote the relative position of the jth stationary beacon observed by the kth sensor expressed in frame S as q_j^k. Thus, in the absence of measurement noise

$$p_j = R(q_j^k + s^k) + b = Ra_j + b, \quad j \in \mathcal{I}(t), \tag{4.25}$$

where $a_j = q_j^k + s^k$ are positions of these points expressed in S. In practice, the a_j is obtained from range measurements that have additive noise; we denote as a_j^m the measured vectors. In the case of lidar range measurements, these are given by

$$a_j^m = (q_j^k)^m + s^k = (\varrho_j^k)^m u^k + s^k, \;\; j \in \mathcal{I}(t), \tag{4.26}$$

where $(\varrho_j^k)^m$ is the measured range to the point by the kth sensor. The mean of the vectors p_j and a_j^m are denoted as $\bar{p}$ and $\bar{a}^m$, respectively, and satisfy

$$\bar{a}^m = R^{\mathrm{T}}(\bar{p} - b) + \bar{\varsigma}, \tag{4.27}$$

where

$$\bar{p} = \tfrac{1}{j}\sum_{j=1}^{j} p_j, \, \bar{a}^m = \tfrac{1}{j}\sum_{j=1}^{j} a_j^m$$

and $\bar{\varsigma}$ is the additive measurement noise obtained by averaging the measurement noise vectors for each of the a_j. Consider the $\binom{j}{2}$ relative position vectors from optical measurements, denoted as $d_j = p\lambda - p\ell$ in frame O and the corresponding vectors in frame S as $l_j = a\lambda - a\ell$, for $\lambda, \ell \in \mathcal{I}(t)$, $\lambda \neq \ell$. The β measured inertial vectors are included in the set of d_j, and their corresponding measured values expressed in frame S are included in the set of l_j. If the total number of measured vectors (both optical and inertial), $\binom{j}{2} + \beta = 2$, then $l_3 = l_1 \times l_2$ is considered a third measured direction in frame S with corresponding vector $d_3 = d_1 \times d_2$ in frame O. Therefore,

$$d_j = Rl_j \Rightarrow D = RL, \tag{4.28}$$

where $D = [d_1 \ldots d_n]$, $L = [l_1 \cdots l_n] \in \mathbb{R}^{3\times n}$ with $n = 3$ if $\binom{j}{2} + \beta = 2$ and $n = \binom{j}{2} + \beta$ if $\binom{j}{2} + \beta > 2$. Note that the matrix D consists of vectors known in frame O. Denote the measured value of matrix L in the presence of measurement noise as L^m. Then,

$$L^m = R^{\mathrm{T}} D + \mathcal{L}, \tag{4.29}$$

where $\mathcal{L} \in \mathbb{R}^{3\times n}$ consists of the additive noise in the vector measurements made in the body frame S.

4.3.2 Velocities Measurement Model

Denote the translational velocity of the rigid body expressed in body fixed frame S by ν. Therefore, one can write the kinematics of the rigid body as

$$\begin{cases} \dot{\Omega} &= R\Omega^{\times}, \\ \dot{b} &= R\nu \end{cases} \Rightarrow \dot{\mathfrak{g}} = \mathfrak{g}\xi^{\vee}, \text{ where } \xi = \begin{bmatrix} \Omega \\ \nu \end{bmatrix} \in \mathbb{R}^6 \text{ and } \xi^{\vee} = \begin{bmatrix} \Omega^{\times} & \nu \\ 0 & 0 \end{bmatrix}. \tag{4.30}$$

For the initial development of the motion estimation scheme, it is assumed that the velocities are directly measured. The estimator is then extended to cover the cases where: (1) only angular velocity is directly measured and (2) none of the velocities are directly measured.

4.3.3 Kinematics of Pose Estimation Error

Denote the estimated pose and its kinematics as

$$\hat{\mathfrak{g}} = \begin{bmatrix} \hat{R} & \hat{b} \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3), \ \dot{\hat{\mathfrak{g}}} = \hat{\mathfrak{g}}\hat{\xi}^{\vee}, \tag{4.31}$$

where $\hat{\xi}$ is rigid body velocities estimate, with $\hat{\mathfrak{g}}_0$ as the initial pose estimate and the pose estimation error as

$$\mathfrak{h} = \mathfrak{g}\hat{\mathfrak{g}}^{-1} = \begin{bmatrix} Q & b - Q\hat{b} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} Q & x \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3), \tag{4.32}$$

where $x = b - Q\hat{b}$. Then one obtains, in the case of perfect measurements,

$$\dot{\mathfrak{h}} = \mathfrak{h}\varphi^{\vee}, \text{ where } \varphi(\hat{\mathfrak{g}}, \xi^m, \hat{\xi}) = \begin{bmatrix} \omega \\ \upsilon \end{bmatrix} = \mathrm{Ad}_{\hat{\mathfrak{g}}}\left(\xi^m - \hat{\xi}\right), \tag{4.33}$$

where $\mathrm{Ad}_{\mathfrak{g}} = \begin{bmatrix} \mathcal{R} & 0 \\ \mathfrak{b}^{\times}\mathcal{R} & \mathcal{R} \end{bmatrix}$ for $\mathfrak{g} = \begin{bmatrix} \mathcal{R} & \mathfrak{b} \\ 0 & 1 \end{bmatrix}$.

4.3.4 Action Functional of the Lagrangian

Consider the sum of rotational and translational measurement residuals between the measurements and estimated pose as a potential energy-like function. The rotational potential function is expressed as Equation 4.8. Consider the translational potential function

$$\mathcal{U}_t(\hat{\mathfrak{g}}, \bar{a}^m, \bar{p}) = \frac{1}{2}\kappa y^{\mathrm{T}} y = \frac{1}{2}\kappa \left\| \bar{p} - \hat{R}\bar{a}^m - \hat{b} \right\|^2, \tag{4.34}$$

where $\bar{p}$ is defined by Equation 4.27, $y \equiv y(\hat{\mathfrak{g}}, \bar{a}^m, \bar{p}) = \bar{p} - \hat{R}\bar{a}^m - \hat{b}$ and κ is a positive scalar. Therefore, the total potential function is defined as the sum of Equation 4.8 for attitude determination on SO(3), and the translational energy (Equation 4.34) as

$$\begin{aligned} \mathcal{U}(\hat{\mathfrak{g}}, L^m, D, \bar{a}^m, \bar{p}) &= \mathcal{U}_r(\hat{\mathfrak{g}}, L^m, D) + \mathcal{U}_t(\hat{\mathfrak{g}}, \bar{a}^m, \bar{p}) \\ &= \Phi\left(\frac{1}{2}\left\langle D - \hat{R}L^m, (D - \hat{R}L^m)W \right\rangle\right) + \frac{1}{2}\kappa \| \bar{p} - \hat{R}\bar{a}^m - \hat{b} \|^2. \end{aligned} \tag{4.35}$$

Define the kinetic energy-like function:

$$\mathcal{T}\left(\varphi(\hat{\mathfrak{g}},\xi^m,\hat{\xi})\right)=\frac{1}{2}\varphi(\hat{\mathfrak{g}},\xi^m,\hat{\xi})^{\mathrm{T}}\mathbb{J}\varphi(\hat{\mathfrak{g}},\xi^m,\hat{\xi}), \tag{4.36}$$

where $\mathbb{J}\in\mathbb{R}^{6\times6}>0$ is an artificial inertia-like kernel matrix. Note that in contrast to rigid body inertia matrix, $\mathbb{J}$ is not subject to intrinsic physical constraints like the triangle inequality, which dictates that the sum of any two eigenvalues of the inertia matrix has to be larger than the third. Instead, $\mathbb{J}$ is a gain matrix that can be used to tune the estimator. For notational convenience, $\varphi(\hat{\mathfrak{g}},\xi^m,\hat{\xi})$ is denoted as φ from now on; this quantity is the velocities estimation error in the absence of measurement noise. Now define the Lagrangian

$$\mathcal{L}(\hat{\mathfrak{g}},L^m,D,\bar{a}^m,\bar{p},\varphi)=\mathcal{T}(\varphi)-\mathcal{U}(\hat{\mathfrak{g}},L^m,D,\bar{a}^m,\bar{p}), \tag{4.37}$$

and the corresponding action functional over an arbitrary time interval $[t_0, T]$ for $T>0$,

$$\mathcal{S}\left(\mathcal{L}(\hat{\mathfrak{g}},L^m,D,\bar{a}^m,\bar{p},\varphi)\right)=\int_{t_0}^{T}\mathcal{L}(\hat{\mathfrak{g}},L^m,D,\bar{a}^m,\bar{p},\varphi)\mathrm{d}t, \tag{4.38}$$

such that $\dot{\hat{\mathfrak{g}}}=\hat{\mathfrak{g}}(\hat{\xi})^{\vee}$. The following statement gives the form of the Lagrangian when perfect (noise-free) measurements are available, and derives the variational estimator for rigid body pose and velocities.

Lemma 4.2

In the absence of measurement noise, the Lagrangian is of the form

$$\mathcal{L}(\mathfrak{h},D,\bar{p},\varphi)=\frac{1}{2}\varphi^T\mathbb{J}\varphi-\Phi\left(\langle I-Q,K\rangle\right)-\frac{1}{2}\kappa y^T y, \tag{4.39}$$

where $K=DWD^T$ *and* $y\equiv y(\mathfrak{h},\bar{p})=Q^T x+(I-Q^T)\bar{p}$.

Proof. Suppose that all the measured states are noise free. Therefore, one can replace $L^m=L$, $\bar{a}^m=\bar{a}$ and $\xi^m=\xi$. The rotational potential function (Equation 4.7) can be replaced by

$$\begin{aligned}\mathcal{U}_r^0(\mathfrak{h},D)&=\frac{1}{2}\left\langle D-\hat{R}L^m,\left(D-\hat{R}L^m\right)W\right\rangle=\frac{1}{2}\left\langle D-Q^TD,\left(D-Q^TD\right)W\right\rangle\\&=\frac{1}{2}\left\langle I-Q^T,\left(I-Q^T\right)DWD^T\right\rangle=\left\langle I-Q,K\right\rangle,\end{aligned} \tag{4.40}$$

since $\hat{R}E=Q^TD$ for the noise-free case. In addition,

$$y(\mathfrak{h},\bar{p})=\bar{p}-\hat{R}\bar{a}^m-\hat{b}=\bar{p}-\hat{R}\bar{a}-\hat{b}=\bar{p}-Q^{\mathrm{T}}R\bar{a}-Q^{\mathrm{T}}(b-x)=Q^{\mathrm{T}}x+(I-Q^{\mathrm{T}})\bar{p}. \tag{4.41}$$

The translational potential function in the absence of measurement noise can be expressed as

$$\mathcal{U}_t(\mathfrak{h},\bar{p}) = \frac{1}{2}\kappa y^{\mathrm{T}} y. \tag{4.42}$$

Therefore, the total potential energy function is

$$\mathcal{U}(\mathfrak{h},D,\bar{p}) = \mathcal{U}_r(\mathfrak{h},D) + \mathcal{U}_t(\mathfrak{h},\bar{p}) = \Phi\left(\mathcal{U}_r^0(\mathfrak{h},D)\right) + \mathcal{U}_t(\mathfrak{h},\bar{p}) = \Phi\left(\langle I-Q,K\rangle\right) + \frac{1}{2}\kappa y^{\mathrm{T}} y, \tag{4.43}$$

and the kinetic energy function is

$$\mathcal{T}(\varphi) = \frac{1}{2}\varphi^{\mathrm{T}}\mathbb{J}\varphi. \tag{4.44}$$

Substituting Equations 4.43 and 4.44 into:

$$\mathcal{L}(\mathfrak{h},D,\bar{p},\varphi) = \mathcal{T}(\varphi) - \mathcal{U}(\mathfrak{h},D,\bar{p}) = \mathcal{T}(\varphi) - \Phi\left(\mathcal{U}_r^0(\mathfrak{h},D)\right) - \mathcal{U}_t(\mathfrak{h},\bar{p}), \tag{4.45}$$

gives the Lagrangian (Equation 4.39) for the noise-free case.

4.3.5 Variational Estimator for Pose and Velocities

The nonlinear variational estimator obtained by applying the Lagrange–d'Alembert principle to the Lagrangian (Equation 4.37), with a dissipation term linear in the velocities estimation error, is given by the following statement. This estimation scheme was shown to be almost globally asymptotically stable in the recent work [4].

Theorem 4.3

The nonlinear variational estimator for pose and velocities is given by

$$\begin{cases} \mathbb{J}\dot{\varphi} & = \mathrm{ad}_{\varphi}^{*}\mathbb{J}\varphi - Z(\hat{\mathfrak{g}},L^m,D,\bar{a}^m,\bar{p}) - \mathbb{D}\varphi, \\ \hat{\xi} & = \xi^m - \mathrm{Ad}_{\hat{\mathfrak{g}}^{-1}}\varphi, \\ \dot{\hat{\mathfrak{g}}} & = \hat{\mathfrak{g}}(\hat{\xi})^{\vee}, \end{cases} \tag{4.46}$$

where $\mathrm{ad}_{\zeta}^{*} = (\mathrm{ad}_{\zeta})^T$ *with* ad_{ζ} *defined by Equation* 4.49, *and* $Z(\hat{\mathfrak{g}},L^m,D,\bar{a}^m,\bar{p})$ *is defined by*

$$Z(\hat{\mathfrak{g}},L^m,D,\bar{a}^m,\bar{p}) = \begin{bmatrix} \Phi'\left(\mathcal{U}_r^0(\hat{\mathfrak{g}},L^m,D)\right)S_{\Gamma}(\hat{R}) + \kappa\bar{p}^{\times}y \\ \kappa y \end{bmatrix}, \tag{4.47}$$

where $\mathcal{U}_r^0(\hat{\mathfrak{g}},L^m,D)$ *is defined as* (*Equation* 4.7), $\mathbb{D}\in\mathbb{R}^{6\times 6}>0$, $y \equiv y(\hat{\mathfrak{g}},\bar{a}^m,\bar{p}) = \bar{p} - \hat{R}\bar{a}^m - \hat{b}$ *and*

$$S_{\Gamma}(\hat{R}) = \mathrm{vex}\left(\Gamma\hat{R}^T - \hat{R}\Gamma^T\right) = \mathrm{vex}\left(DW(L^m)^T\hat{R}^T - \hat{R}L^mWD^T\right). \tag{4.48}$$

Proof. A Rayleigh dissipation term linear in the velocities of the form $\mathbb{D}\varphi$ is used in addition to the Lagrangian (Equation 4.39), and the Lagrange–d'Alembert principle from variational mechanics is applied to obtain the estimator on TSE(3). *Reduced variations* with respect to h and φ [7] are applied, given by

$$\delta\mathfrak{h} = \mathfrak{h}\eta^{\vee}, \delta\varphi = \dot{\eta} + \mathrm{ad}_{\varphi}\eta, \text{ where } \eta^{\vee} = \begin{bmatrix} \Sigma^{\times} & \rho \\ 0 & 0 \end{bmatrix} \text{ and } \mathrm{ad}_{\zeta} = \begin{bmatrix} w^{\times} & 0 \\ v^{\times} & w^{\times} \end{bmatrix}, \tag{4.49}$$

for $\eta = \begin{bmatrix} \Sigma \\ \rho \end{bmatrix} \in \mathbb{R}^6$ and $\zeta = \begin{bmatrix} w \\ v \end{bmatrix} \in \mathbb{R}^6$, with $\eta(t_0) = \eta(T) = 0$. This leads to the expression:

$$\delta_{\mathrm{h},\varphi}\mathcal{S}\left(\mathcal{L}(\mathfrak{h}, D, \bar{p}, \varphi)\right) = \int_{t_0}^{T} \eta^{\mathrm{T}}\mathbb{D}\varphi \mathrm{d}t. \tag{4.50}$$

Note that the variations of the attitude and position estimation errors are of the form

$$\delta Q = Q\Sigma^{\times}, \delta x = Q\rho, \tag{4.51}$$

respectively. Applying reduced variations to the rotational potential energy term (Equation 4.40), one obtains

$$\delta_Q \mathcal{U}_r^0(\mathfrak{h}, D) = \langle -Q\Sigma^{\times}, K\rangle = \frac{1}{2}\left\langle \Sigma^{\times}, KQ - Q^{\mathrm{T}}K \right\rangle = S_K^{\mathrm{T}}(Q)\Sigma, \tag{4.52}$$

where

$$S_K(Q) = \mathrm{vex}\left(KQ - Q^{\mathrm{T}}K\right). \tag{4.53}$$

Taking first variation of the translational potential energy term (Equation 4.42) with respect to Q and x yields:

$$\delta_{\mathfrak{h}}\mathcal{U}_t(\mathfrak{h}, \bar{p}) = \kappa(\delta x + \delta Q\bar{p})^{\mathrm{T}}\left\{x + (Q - I)\bar{p}\right\} = \kappa\left(\rho^{\mathrm{T}}y + \Sigma^{\mathrm{T}}\bar{p}^{\times}y\right). \tag{4.54}$$

Therefore, the first variation of Equation 4.43 with respect to estimation errors is

$$\delta_{\mathfrak{h}}\mathcal{U}(\mathfrak{h}, D, \bar{p}) = Z^{\mathrm{T}}(\mathfrak{h}, D, \bar{p})\eta, \tag{4.55}$$

where $Z(\mathfrak{h}, D, \bar{p})$ is defined by

$$Z(\mathfrak{h}, D, \bar{p}) = \begin{bmatrix} \Phi'\left(\langle I - Q, K\rangle\right)S_K(Q) + \kappa\bar{p}^{\times}\left\{Q^{\mathrm{T}}x + (I - Q^{\mathrm{T}})\bar{p}\right\} \\ \kappa\{Q^{\mathrm{T}}x + (I - Q^{\mathrm{T}})\bar{p}\} \end{bmatrix}. \tag{4.56}$$

Taking the first variation of the kinetic energy term (Equation 4.44) with respect to ϕ results in

$$\delta_{\varphi}\mathcal{T}(\varphi) = \varphi^{\mathrm{T}}\mathbb{J}\delta\varphi = \varphi^{\mathrm{T}}\mathbb{J}(\dot{\eta} + \mathrm{ad}_{\varphi}\eta), \tag{4.57}$$

applying the reduced variation for $\delta\varphi$ as given in Equation 4.49. Therefore, the first variation of the action functional (Equation 4.38) is obtained as

$$\begin{aligned}\delta_{\mathfrak{h},\varphi}\mathcal{S}\big(\mathcal{L}(\mathfrak{h},D,\bar{p},\varphi)\big) &= \int_{t_0}^{T}\Big\{\varphi^{\mathrm{T}}\mathbb{J}(\dot{\eta}+\mathrm{ad}_{\varphi}\eta)-\eta^{\mathrm{T}}Z(\mathfrak{h},D,\bar{p})\Big\}\mathrm{d}t\\ &= \int_{t_0}^{T}\eta^{\mathrm{T}}\Big(\mathrm{ad}_{\varphi}^{*}\mathbb{J}\varphi-Z(\mathfrak{h},D,\bar{p})-\mathbb{J}\dot{\varphi}\Big)\mathrm{d}t+\varphi^{\mathrm{T}}\mathbb{J}\eta\Big|_{t_0}^{T}\\ &= \int_{t_0}^{T}\eta^{\mathrm{T}}\Big(\mathrm{ad}_{\varphi}^{*}\mathbb{J}\varphi-Z(\mathfrak{h},D,\bar{p})-\mathbb{J}\dot{\varphi}\Big)\mathrm{d}t,\end{aligned} \tag{4.58}$$

applying fixed endpoint variations with $\eta(t_0)=\eta(T)=0$. By substituting Equation 4.58 in Equation 4.50, one obtains

$$\mathbb{J}\dot{\varphi}=\mathrm{ad}_{\varphi}^{*}\mathbb{J}\varphi-Z(\mathfrak{h},D,\bar{p})-\mathbb{D}\varphi. \tag{4.59}$$

In order to implement this estimator using the aforementioned measurements, substitute $Q^{\mathrm{T}}D=\hat{R}L^{m}$. This changes the rotational potential energy formed by the estimation errors in attitude (Equation 4.40) to Equation 4.7. Equation 4.53 is also reformulated as

$$S_K(Q)=\mathrm{vex}(DWD^{\mathrm{T}}Q-Q^{\mathrm{T}}DWD^{\mathrm{T}})=\mathrm{vex}(DW(L^m)^{\mathrm{T}}\hat{R}^{\mathrm{T}}-\hat{R}(L^m)WD^{\mathrm{T}})=S_{\Gamma}(\hat{R}). \tag{4.60}$$

Finally, the second row in the matrix $Z(\mathrm{h},D,\bar{p})$ is replaced by

$$\begin{aligned}\kappa\{Q^{\mathrm{T}}x+(I-Q^{\mathrm{T}})\bar{p}\} &= \kappa\{Q^{\mathrm{T}}b-\hat{b}+\bar{p}-Q^{\mathrm{T}}\bar{p}\}=\kappa\{\hat{R}R^{\mathrm{T}}(b-\bar{p})-\hat{b}+\bar{p}\}\\ &= \kappa\{-\hat{R}\bar{a}^{m}-\hat{b}+\bar{p}\}.\end{aligned} \tag{4.61}$$

Taking these changes into account, one could obtain the first of equations (Equation 4.46). Thus, the complete nonlinear estimator equations are given by (4.46).

Explicit expressions for the vector of velocities ξ^m can be obtained for two common cases when these velocities are not directly measured. These two cases are dealt with in Section 4.3.6.

4.3.6 Variational Estimator Implemented without Direct Velocity Measurements

The velocity measurements in Equation 4.46 can be replaced by filtered velocity estimates obtained by linear filtering of optical and inertial measurements using, for example, a second-order Butterworth filter. This is both useful and necessary when velocities are not directly measured. The filtered values ξ^f are then used in place of ξ^m to enhance the nonlinear estimator given by Theorem 4.3. Denote the measured vector quantity at time t by z^m. A linear second-order filter of the following form is used:

$$\ddot{z}^{f}+2\mu\omega_n\dot{z}^{f}=\omega_n^2\left(z^m-z^f\right), \tag{4.62}$$

where:

- ω_n is the natural (cutoff) frequency
- μ is the damping ratio
- z^f is the filtered value of z^m

Thereafter, z^f is used in place of z^m in Equation 4.46.

4.3.6.1 Angular Velocity is Measured Using Rate Gyros

For the case that rate gyro measurements of angular velocities are available besides the j feature point (or beacon) position measurements, the linear velocities of the rigid body can be calculated using each single position measurement by rewriting Equation 4.65 as

$$\nu^f = (a_j^f)^\times \Omega^f - v_j^f, \tag{4.63}$$

for the jth point. Averaging the values of ν derived from all feature points gives a more reliable result. Therefore, the rigid body's filtered velocities are expressed in this case as

$$\xi^f = \begin{bmatrix} \Omega^f \\ \dfrac{1}{j}\displaystyle\sum_{j=1}^{j} (a_j^f)^\times \Omega^f - v_j^f \end{bmatrix}. \tag{4.64}$$

4.3.6.2 Translational and Angular Velocity Measurements are not Available

In the case that both angular and translational velocity measurements are not available or accurate, rigid body velocities can be calculated in terms of the inertial and optical measurements. In order to do so, one can differentiate Equation 4.25 as follows:

$$\begin{gathered} \dot{p}_j = R\Omega^\times a_j + R\dot{a}_j + \dot{b} = R\left(\Omega^\times a_j + \dot{a}_j + \nu\right) = 0 \Rightarrow \dot{a}_j - a_j^\times \Omega + \nu = 0 \\ \Rightarrow v_j = \dot{a}_j = [a_j^\times \;\; -I]\xi = G(a_j)\xi, \end{gathered} \tag{4.65}$$

where $G(a_j) = [a_j^\times \; -I]$ has full row rank. From vision-based or Doppler lidar sensors, one can also measure the velocities of the observed points in frame S, denoted v_j^m. Here, velocity measurements as would be obtained from vision-based sensors are considered. The measurement model for the velocity is of the form

$$v_j^m = G(a_j)\xi + \vartheta_j, \tag{4.66}$$

where $\vartheta_j \in \mathbb{R}^3$ is the additive error in velocity measurement v_j^m. Note that $v_j = \dot{a}_j$, for $j \in \mathcal{I}(t)$. As this kinematics indicates, the relative velocities of at least three beacons are needed to determine the vehicle's translational and angular velocities uniquely at each instant. However, when only one or two landmarks/beacons are measured, the estimator can propagate velocity estimates based on a least squares velocity determined from the available measurements. The rigid body velocities in both cases are obtained using the pseudo-inverse of $\mathbb{G}(A^f)$:

$$\mathbb{G}(A^f)\xi^f = \mathbb{V}(V^f) \Rightarrow \xi^f = \mathbb{G}^\ddagger(A^f)\mathbb{V}(V^f), \text{ where } \mathbb{G}(A^f) = \begin{bmatrix} G(a_1^f) \\ \vdots \\ G(a_j^f) \end{bmatrix} \text{and } \mathbb{V}(V^f) = \begin{bmatrix} v_1^f \\ \vdots \\ v_j^f \end{bmatrix}, \tag{4.67}$$

for $1,\ldots,j \in \mathcal{I}(t)$. When at least three beacons are measured, $\mathbb{G}(A^f)$ is a full column rank matrix, and $\mathbb{G}^\ddagger(A^f) = \left(\mathbb{G}^{\mathrm{T}}(A^f)\mathbb{G}(A^f)\right)^{-1}\mathbb{G}^{\mathrm{T}}(A^f)$ gives its pseudo-inverse. For the case that only one or two beacons are observed, $\mathbb{G}^{\mathrm{T}}(A^f)$ is a full row rank matrix, whose pseudo-inverse is given by $\mathbb{G}^\ddagger(A^f) = \mathbb{G}^{\mathrm{T}}(A^f)\left(\mathbb{G}(A^f)\mathbb{G}^{\mathrm{T}}(A^f)\right)^{-1}$.

4.3.7 Discrete-Time Variational Pose Estimator

Let $(\hat{\mathfrak{g}}_i, \hat{\xi}_i) \in \mathrm{SE}(3) \times \mathbb{R}^6$ denote the discrete state estimate at time t_i, such that $(\hat{\mathfrak{g}}_i, \hat{\xi}_i) \approx (\hat{\mathfrak{g}}(t_i), \hat{\xi}(t_i))$ where $(\hat{\mathfrak{g}}(t), \hat{\xi}(t))$ is the exact solution of the continuous-time estimator at time $t \in [t_0, T]$. Let the values of the discrete-time measurements ξ^m, $\bar{a}^m$ and L^m at time t_i be denoted as ξ_i^m, $\bar{a}_i^m$, and L_i^m, respectively. Furthermore, denote the corresponding values for the latter two quantities in inertial frame at time t_i by $\bar{p}_i$ and D_i, respectively.

Theorem 4.4

A first-order discretization of the estimator proposed in Theorem 4.3 is given by

$$(J\omega_i)^\times = \frac{1}{\Delta t}(F_i\mathcal{J} - \mathcal{J}F_i^T), \tag{4.68}$$

$$(M + \Delta t\mathbb{D}_t)\upsilon_{i+1} = F_i^T M\upsilon_i + \Delta t\kappa(\hat{b}_{i+1} + \hat{R}_{i+1}\bar{a}_{i+1}^m - \bar{p}_{i+1}), \tag{4.69}$$

$$\begin{aligned}(J + \Delta t\mathbb{D}_r)\omega_{i+1} = F_i^T J\omega_i + \Delta t M\upsilon_{i+1}\times\upsilon_{i+1} + \Delta t\kappa\bar{p}_{i+1}^\times(\hat{b}_{i+1} + \hat{R}_{i+1}\bar{a}_{i+1}^m)\\ -\Delta t\Phi'\Big(\mathcal{U}_r^0(\hat{\mathfrak{g}}_{i+1}, L_{i+1}^m, D_{i+1})\Big)S_{\Gamma_{i+1}}(\hat{R}_{i+1}),\end{aligned} \tag{4.70}$$

$$\hat{\xi}_i = \xi_i^m - \mathrm{Ad}_{\hat{\mathfrak{g}}_i^{-1}}\varphi_i, \tag{4.71}$$

$$\hat{\mathfrak{g}}_{i+1} = \hat{\mathfrak{g}}_i\exp(\Delta t\hat{\xi}_i^\vee), \tag{4.72}$$

where M, J *are positive definite matrices and* $\mathcal{J}$ *is defined in terms of the matrix* J *by* $\mathcal{J} = 1/2\ \mathrm{trace}\,[J]I - J$, $F_i \in \mathrm{SO(3)}$, $\big(\hat{\mathfrak{g}}(t_0), \hat{\xi}(t_0)\big) = (\hat{\mathfrak{g}}_0, \hat{\xi}_0)$, $\varphi_i = [\omega_i^T\ \upsilon_i^T]^T$, *and* $S_{\Gamma_i}(\hat{R}_i)$ *is the value of* $S_\Gamma(\hat{R})$ *at time* t_i, *with* $S_\Gamma(\hat{R})$ *as defined by* Equation 4.48.

Remark 4.1

In the absence of any direct velocity measurements or only angular velocity measurements, the expressions provided in Section 3.6 to calculate rigid body velocities are still valid in discrete time. One can use the discrete-time variables introduced in this section in place of their continuous-time counterparts. The second-order Butterworth filter (*Equation* 4.62) *is discretized using the Newmark-*β *method as follows:*

$$\begin{cases} z_{i+1}^f = z_i^f + \Delta t\dot{z}_i^f + \dfrac{\Delta t^2}{4}(\ddot{z}_i^f + \ddot{z}_{i+1}^f) \\ \dot{z}_{i+1}^f = \dot{z}_i^f + \dfrac{\Delta t}{2}(\ddot{z}_i^f + \ddot{z}_{i+1}^f) \end{cases}. \tag{4.73}$$

This method gives the filtered positions and velocities as follows:

$$\begin{aligned}\begin{Bmatrix} z_{i+1}^f \\ \dot{z}_{i+1}^f \end{Bmatrix} = \frac{1}{4 + 4\mu\omega_n\Delta t + \omega_n^2\Delta t^2}\\ \begin{bmatrix} 4 + 4\mu\omega_n\Delta t - \omega_n^2\Delta t^2 & 4\Delta t & \omega_n^2\Delta t^2 \\ -4\omega_n^2\Delta t & 4 - 4\mu\omega_n\Delta t - \omega_n^2\Delta t^2 & 2\omega_n^2\Delta t \end{bmatrix}\begin{Bmatrix} z_i^f \\ \dot{z}_i^f \\ z_i^m + z_{i+1}^m \end{Bmatrix},\end{aligned} \tag{4.74}$$

where z_i^m and z_i^f are the corresponding value of quantities z^m and z^f at time instant t_i, respectively. As with the continuous time version, ξ_i^m can be replaced with ξ_i^f in the estimator equations in the absence of velocity measurements.

4.3.8 Numerical Simulations

This section presents numerical simulation results for the discrete-time estimator obtained in Section 3.7. In order to numerically simulate this estimator, simulated true states of an aerial vehicle flying in a cubical volume are produced using the kinematics and dynamics equations of a rigid body. The vehicle mass and body inertia are taken to be m_v = 420 g and $J_v = [51.2 \quad 60.2 \quad 59.6]^{\mathrm{T}}$ g.m², respectively. The resultant external forces and torques applied to the vehicle are $\phi_v(t) = 10^{-3}[10\cos(0.1t)\;\; 2\sin(0.2t)\;\; -2\sin(0.5t)]^{\mathrm{T}}$ N and $\tau_v(t) = 10^{-6}\phi_v(t)$ N.m, respectively. The volume is assumed to be a cube of size 10 m × 10 m × 10 m with the inertial frame origin at its geometric center. The initial attitude and position of the vehicle are as follows:

$$R_0 = \mathrm{expm}_{\mathrm{SO(3)}}\left(\left[\frac{\pi}{4}\times\left(\frac{3}{7}\;-\frac{6}{7}\;\frac{2}{7}\right)^{\mathrm{T}}\right]^{\times}\right), \text{ and } b_0 = [2.5\; 0.5\; -3]^{\mathrm{T}} \text{ m}. \tag{4.75}$$

The vehicle's initial angular and translational velocities are, respectively:

$$\Omega_0 = [0.2\;\; -0.05\;\; 0.1]^{\mathrm{T}} \text{ rad/s and } \nu_0 = [-0.05\;\; 0.15\;\; 0.03]^{\mathrm{T}} \text{ m/s} \tag{4.76}$$

The vehicle dynamics is simulated over a time interval of T = 150 s, with a time stepsize of Δt = 0.02 s. The trajectory of the vehicle over this time interval is depicted in Figure 4.4. The following two inertial directions, corresponding to nadir and earth's magnetic field directions, are measured by the inertial sensors on the vehicle:

$$d_1 = [0\;\; 0\;\; -1]^{\mathrm{T}}, \quad d_2 = [0.1\;\; 0.975\;\; -0.2]^{\mathrm{T}}. \tag{4.77}$$

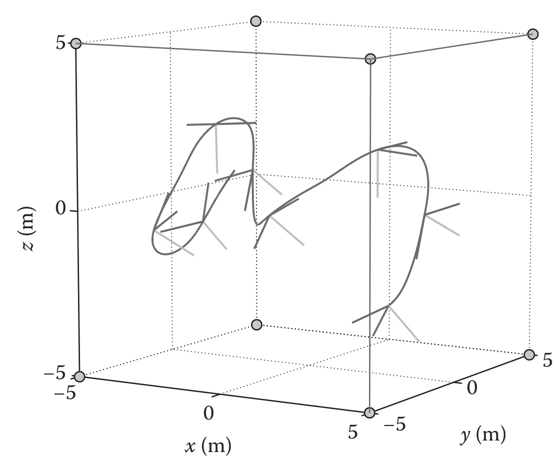

FIGURE 4.4 Position and attitude trajectory of the simulated vehicle.

For optical measurements, eight beacons are located at the eight vertices of the cube, labeled 1 to 8. The positions of these beacons are known in the inertial frame and their index (label) and relative positions are measured by optical sensors onboard the vehicle whenever the beacons come into the field of view (FOV) of the sensors. Three identical cameras (optical sensors) and inertial sensors are assumed to be installed on the vehicle. The cameras are fixed to known positions on the vehicle, on a hypothetical horizontal plane passing through the vehicle, 120° apart from each other, as shown in Figure 4.3. All the camera readings contain random zero mean signals whose probability distributions are normalized bump functions with width of 0.001 m. The following are selected for the positive definite estimator gain matrices:

$$\begin{aligned} J &= \mathrm{diag}\big([0.9\ \ 0.6\ \ 0.3]\big), \quad M = \mathrm{diag}\big([0.0608\ \ 0.0486\ \ 0.0365]\big), \\ \mathbb{D}_r &= \mathrm{diag}\big([2.7\ \ 2.2\ \ 1.5]\big), \quad \mathbb{D}_t = \mathrm{diag}\big([0.1\ \ 0.12\ \ 0.14]\big). \end{aligned} \tag{4.78}$$

Similar to Section 2.7, $\Phi(\mathcal{X}) = \mathcal{X}$ and the initial state estimates have the following values:

$$\hat{\mathrm{g}}_0 = I,\ \ \hat{\Omega}_0 = [0.1\ \ 0.45\ \ 0.05]^{\mathrm{T}}\ \text{rad/s, and}\ \hat{\mathrm{v}}_0 = [2.05\ \ 0.64\ \ 1.29]^{\mathrm{T}}\ \text{m/s.} \tag{4.79}$$

A conic FOV of 2 × 40° is assumed for the cameras, which guarantees at least three beacons observed are common between successive measurements. The vehicle's velocity vector is calculated from Equation 4.67. The discrete-time estimator (Equations 4.68 through 4.72) is simulated over a time interval of $T = 20$ s with time stepsize $\Delta t = 0.02$ s. At each measurement instant, (4.68) is solved using Newton–Raphson iterations to find an approximation for F_i. The remaining equations (all explicit) are solved consecutively to generate the estimated states. The principal angle of the attitude estimation error and the position estimate error are plotted in Figure 4.5. The components of the vehicle's velocity estimate errors are also depicted in Figure 4.6. All estimation errors are shown to converge to a neighborhood of $(\mathrm{h}, \varphi) = (I, 0)$, where the size of this neighborhood depends on the magnitude of measurement noise.

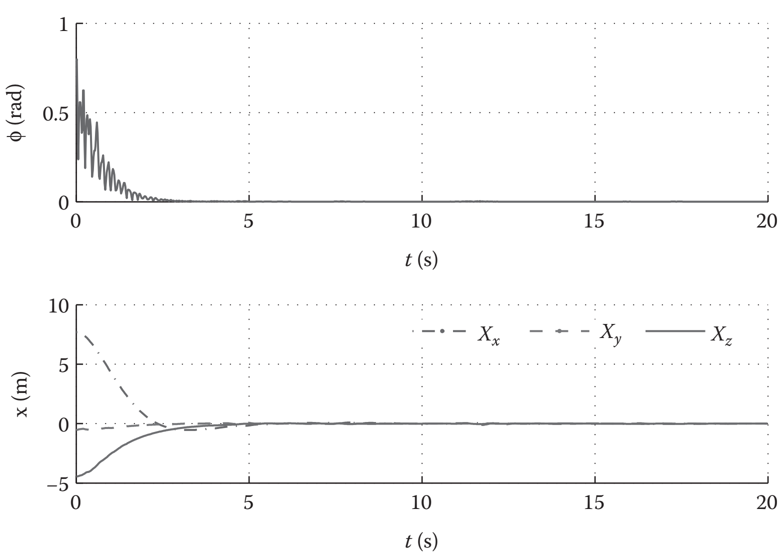

FIGURE 4.5 Angular and translational velocity estimation error.

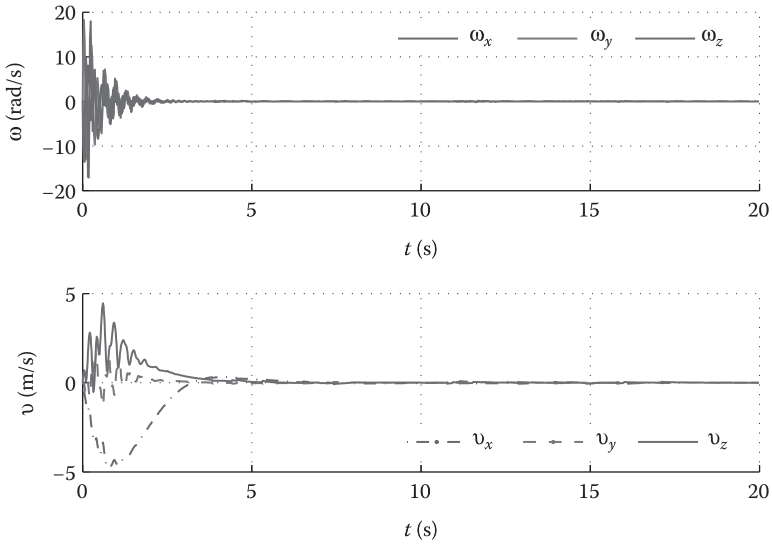

FIGURE 4.6 Principal angle of the attitude and position estimation error.

4.4 CONCLUSION

This chapter describes the variational state estimation schemes for rigid body rotational and translational motion, assuming measurements from body-fixed sensors. The sensors are assumed to provide measurements in continuous-time or at a sufficiently high frequency, with bounded measurement noise. An artificial kinetic energy quadratic in velocity estimation errors is defined, as well as artificial potential function(s): (1) a generalization of Wahba's cost function for attitude estimation error, which is in the form of a Morse function on the Lie group of rigid body rotations SO(3), and (2) a quadratic function of the vehicle's position estimation error. By applying the Lagrange–d'Alembert principle on a Lagrangian consisting of these energy-like terms and a dissipation term linear in the velocity estimation errors, estimators are designed on the Lie groups of rigid body motions SO(3) and SE(3). The continuous estimators for rotational motion only and coupled rotational and translational motions are discretized by applying the discrete Lagrange–d'Alembert principle to the discrete Lagrangian with dissipation terms linear in the velocity estimation errors. In the presence of measurement noise, numerical simulations show that state estimates converge to a bounded neighborhood of the true states.

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